Universidad Autónoma de Madrid

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Tesis Doctoral

The characterization of the supersymmetric solutions of Supergravity in four and five dimensions

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Dedico este trabajo

a mi esposa e hijas

y

a mi madre

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Resumen

En este trabajo se presenta la caracterización de todas las configuraciones supersimétricas de la Supergravedad $N = 1$, $d = 5$, sin gaugear y gaugeada, acoplada a supermultipletes vectoriales e hipermultipletes, así como también de la Supergravedad pura $N = 4$, $d = 4$. Se usa principalmente el método de los bilineales de espinores para extraer toda la información de las ecuaciones de espinores de Killing. Se determina cómo las identidades de espinores de Killing pueden ser usadas para establecer relaciones entre las ecuaciones de movimiento cuando éstas se evalúan en configuraciones supersimétricas.

Adicionalmente, usando la caracterización conocida de las soluciones supersimétricas de la Supergravedad $N = 2$, $d = 4$ acoplada a supermultipletes vectoriales, se muestra cómo el requisito de existencia de supersimetría en todos los lugares, incluidas aquellos en donde se encuentren las fuentes, puede ser usado como censor cósmico.

Conclusiones

Hemos completado la caracterización de las soluciones supersimétricas de la Supergravedad general $N = 1$, $d = 5$ acoplada a materia y de la Supergravedad pura $N = 4$, $d = 4$. Para tal fin, hemos usado el método de los bilineales de espinores. Como se acostumbra, usando este método uno separa las soluciones entre el caso tipo tiempo y el caso nulo. En el caso tipo tiempo típicamente hay solitones masivos mientras que en el caso nulo hay *pp-waves*.

La Supergravedad $N = 1$, $d = 5$ había sido estudiada antes por varios autores. Nosotros hemos presentado el primer análisis completo con hiperescalares (el estudio de la teora con hiperescalares fue iniciado en Refs. [1, 2]). En el caso tipo tiempo, la principal novedad debida a la presencia de hiperescalares es el agrandamien-to del grupo de holonomía de la variedad base espacial desde $SU(2)$ hasta el grupo completo $SO(4)$, estando la componente anti-autodual de la conexión de espín relacionada a los otros campos. De hecho, en el caso sin gaugear es justo el pull-back de la conexión $\mathfrak{su}(2)$ de la variedad Kähler cuaterniónica (la misma relación se mantiene en el caso gaugeado, pero con algunas correciones). La condición sobre los hiperescalares para tener supersimetría no rota tiene una forma muy simple y sugestiva, de hecho en el caso sin gaugear es la ecuación para mapas cuaterniónicos entre variedades hiperKähler (aunque la variedad base no es necesariamente hiperKähler). Debido a su simplicidad, esta ecuación podría ser el punto de partida para encontrar nuevas soluciones concretas de la teoría.

Previamente no habían análisis sobre la caracterización de las soluciones supersimétricas de la Supergravedad $N = 1$, $d = 5$ acoplada a materia que pertenezcan a la clase nula. Hemos encontrado que en este caso la conexión de espín del subespacio tridimensional transverso a la onda también está relacionada a los otros campos. En el caso sin gaugear también viene dada por el pullback de la conexión $\mathfrak{su}(2)$ de la variedad Kähler cuaterniónica. Igualmente, la condición sobre los hiperescalares es bastante simple.

Encontramos, en una forma muy precisa, las proyecciones genéricas que deben ser impuestas sobre los espinores de Killing para tener supersimetría no rota. En el caso tipo tiempo de esta teoría todas las configuraciones supersimétricas preservan al menos 1/8 de las supersimetrías.

Hemos encontrado soluciones con una isometría adicional en el caso tipo tiempo las cuales son la generalización de la métrica instantónica de Gibbons-Hawking. Como mencionamos, la presencia de hiperescalares destruye la autodualidad de la conexión, este hecho se refleja en la no-trivialidad de la conexión tridimensional, a diferencia del instanton de Gibbons-Hawking el cual tiene la métrica tridimensional plana.

Sería interesante estudiar el mecanismo de atractor y la entropía de las soluciones del tipo agujero negro en presencia de hiperescalares. Más aún, la pp-wave de las soluciones de clase nula se puede reducir dimensionalmente a agujeros negros supersimétricos $N = 2$, $d = 4$. Esto abre nuevas preguntas acerca de cómo el mecanismo de atractor 4-dimensional se implementa en una configuración 5-dimensional, teniendo en cuenta que estas soluciones 5-dimensionales pertenecen a la clase nula y el mecanismo de atractor estándar está demostrado sólo para soluciones de la clase tipo tiempo. El origen 5-dimensional de la entropía 4-dimensional puede (y debe) ser investigada.

Más aún, se puede realizar la reducción dimensional de todas las soluciones supersimétricas 5-dimensionales a 4 dimensiones. Sería interesante ver cómo esto se puede hacer en el contexto de la caracterización de las soluciones supersimétricas (tal caracterización para la teoría $N = 2$, $d = 4$ acoplada a materia se ha hecho en las Refs. [3,4]). Además, la reducción/elevación de las soluciones supersimétricas se puede analizar junto con la teoría seis-dimensional. Por lo tanto, las teorías con $N = 2$ supersimetrías en seis, cinco y cuatro dimensiones pueden ser analizadas en una forma unificada.

En la teoría $N = 4$, $d = 4$ hemos determinado (en el caso tipo tiempo) la forma precisa en la cual las tres clases de holonomía de la variedad base (plana, $U(1)$ y $SU(2)$) aparecen, es decir, hemos indicado cómo la conexión de espín se relaciona a las otras variables. Hemos extendido así el trabajo de Tod [5] quien caracterizó sólo aquellas soluciones con holonomía plana en el espacio base. Aquí, las simetrías de la teoría ($SU(4)$ y $SL(2, \mathbb{R})$) juegan un papel central, guiándonos en la búsqueda de las configuraciones y soluciones supersimétricas.

La metodología que hemos desarrollado para analizar la teoría $N = 4$, $d = 4$ se puede adaptar a otras teorías cuadridimensionales con más supersimetrías, esto es, Supergravedad cuadridimensional $N = 6$ y 8.

Otra continuación interesante de nuestro trabajo sería desarrollar la caracterización con correcciones del tipo R^2 tanto en cuatro como en cinco dimensiones. Esto es particularmente viable ya que las variaciones de supersimetría se mantienen iguales cuando las correcciones del tipo R^2 son tomadas en cuenta (aunque las ecuaciones de movimiento cambian).

Hemos visto que las Identidades de Espinores de Killing (KSIs, por sus siglas en Inglés) generales halladas en Ref. [6] pueden ser usadas para obtener relaciones útiles entre las ecuaciones de movimiento evaluadas en configuraciones supersimétricas. Esta es una herramienta muy poderosa, nos ha permitido por ejemplo evitar la evaluación de (algunas de las componentes de) las Ecuaciones de Einstein. Más aún, las KSIs pueden ser calculadas para cualquier teoría de supergravedad. Otros autores han usado relaciones análogas entre la ecuaciones de movimiento, pero las habían encontrado usando directamente las condiciones de integrabilidad de las ecuaciones de espinores de Killing, la cual es una ruta más difícil que las KSIs

Hemos demostrado además cómo la supersimetría actúa como censor cósmico. Exigiendo que la supersimetría se preserve en todos los lugares, incluyendo las fuentes, las configuraciones se restringen de tal forma que muchas soluciones con patologías (singularidades desnudas) se pueden descartar. Hemos formulado la condición de tener supersimetría preservada en todos los lugares por medio de tres condiciones que los agujeros negros supersimétricos tienen que satisfacer. Hemos demostrado cómo estas condiciones restringen las posibles fuentes debido a, básicamente, la exclusión de aquellas con carga NUT, momento angular, energía negativa y pelo escalar, lo cual aparentemente no puede ser descrito en la Teoría de Cuerdas. Llegamos a una situación en la cual si un observador lejos de una de las configuraciones globalmente supersimétricas que hemos considerado, detecta momento angular y campos escalares no triviales, sólo encontrará fuentes electromagnéticas estáticas en equilibrio cuando se acerque al sistema.

Estas condiciones deberían ser mejoradas al considerar correcciones cuánticas. Otra línea de acción interesante sería considerar la regularidad de las soluciones del tipo agujero negro en teorias con $N > 2$, ver por ejemplo Refs. [7–9], e investigar el papel que juega el atractor [10].

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Introduction

Many of us are inclined to the idea that the fundamental theories describing our world are actually low-energy effects of still more fundamental theories.

This philosophy has been rewarded historically in science. For instance, we may mention a crucial example in high-energy Physics. Fermi invented a model to describe, within the framework of Quantum Field Theory, the weak force responsible of β-decays. Fermi's model did explain the experimental data well, but it is non-renormalizable. This implies that the theory becomes inconsistent at high energies. For some people this was a sign that something was beyond the Fermi's model. Eventually, the theory of electroweak interactions came.

Now String Theory could be beyond the Standard Model of Particles, providing also a quantum dynamics for gravitation. It is the most promising candidate for a theory of everything giving a consistent unification of all the fundamental interactions. Moreover, we could be near to a really big revolution in science if evidence for supersymmetry -a key ingredient of String Theory is found in the newest experiments at high energies.

This thesis is devoted to one topic of String Theory: The characterization of supersymmetric solutions of some theories of supergravities, which are classical effective dynamics for strings. The supersymmetric states of the effective theory are of particular relevance because supersymmetry protects them to get quantum corrections. Hence they are essentially vacuum states of String Theory.

We wish to situate this thesis in the framework of fundamental theoretical Physics. To this end it is worth making a rough overview of the fundamental theories of nature, the Standard Model of Particles and General Relativity, and, next, of String Theory followed by its classical effective theory, Supergravity.

1.1 Our two fundamental theories: The Standard Model and General Relativity

1.1.1 Basis of the Standard Model

The need of Quantum Field Theory

In the early years of twentieth century two major scientific achievements were done: Quantum Mechanics and the Einstein's Theory of Special Relativity. Special Relativity is restricted to the kinematics of non-accelerated movement. Later on, Einstein realized the deep connection between acceleration and gravitation arriving at the Theory of General Relativity.

These revolutionary ideas constitute the basis of Modern Physics. Quantum Mechanics opened the doors of a world much richer than the classical Newton's Laws of mechanics. In particular, Quantum Mechanics brought us to a probabilistic interpretation of fundamental laws rather than the deterministic philosophy of the Newton's mechanics. At the same time, Special Relativity enhanced the symmetry concepts of Galilean relativity by putting time on the same footing of space.

Quantum Mechanics does not possess the space-time symmetry features dictated by Special Relativity. The Schrödinger Equation is not relativistically invariant. Therefore Quantum Mechanics should be modified in some way in order to make it a relativistic theory.

Klein and Gordon as well as Dirac made the pioneering works in Relativistic Quantum Mechanics. Klein and Gordon studied an equation for spinless particles whereas Dirac made an equation for particles with spin $\frac{1}{2}$, like the electron. These theories came with a intriguing feature: the absence of a lower bound for the particle energy.

For the case of the electron, the problem of the unbounded energy is solved by the existence of the so called *Dirac sea*. This consists of the filling of the negative branch of the energy by electrons, avoiding the infinite decay due to the Exclusion Principle.

This leads to the existence of antiparticles. An electron in the Dirac sea can go up to the positive sector by absorbing a photon. The lifted particle leaves a hole, which is nothing but a particle of the same mass and opposite charge of the electron. This kind of particles are called antiparticles and predicting them was one of the most important success of Dirac's theory.

Now we meet with other question: particles can be created from the vacuum. This leads us to conclude that Relativistic Quantum Mechanics is inconsistent because it is a mechanical theory of a single particle.

Quantum Mechanics, in the Heisenberg picture, is formulated in terms of operators which depend on time and whose eigenvalues are physical observables. By analogy, the relativistic extension of Quantum Mechanics must be based on operators whose expectation values are probabilities of finding particles. Thus, the number of particles is not a conserved quantity.

Special Relativity adds one more key ingredient. Operators that only depend on time violate causality. The appropriated operators depend both on space and time and are subject to commutation relations with causal structure.

An operator which depends on space and time is called a *quantum field*. Thus the union of Quantum Mechanics and Special Relativity leads to Quantum Field Theory (QFT).

The minimum scale at which QFT effects can be appreciated on the dynamics of a particle, for instance for the particle/antiparticle pair creation, is given by the Compton wavelength

$$\lambda = m^{-1}h/c \tag{1.1}$$

where m is the mass of the particle.

Quantum Electrodynamics

Maxwell's Electromagnetism, formulated in the last half of the nineteenth century, provided an unified frame for the laws of electricity and magnetism. With it, Maxwell could show that electric and magnetic fields travel in space as waves.

Electromagnetism is a theory of classical fields. It can be formulated in terms of the field strengths which are precisely the electric and magnetic fields. These fields can be arranged into a unique relativistically covariant object $F_{\mu\nu}$, where μ and ν are space-time indices. The electromagnetic field interacts with charged particles obeying classical field equations.

The theory can be alternatively formulated in terms of the vector potential, A_μ, whose derivatives yield the field strength. However, in classical electromagnetism the vector potential is just a mathematical tool with no physical significance by itself.

The formulation of Maxwell's Electromagnetism in terms of the potentials leads to the concept of gauge invariance. Two different potentials A_μ and A'_μ related by

$$A'_\mu = A_\mu + \partial_\mu \Lambda \,, \tag{1.2}$$

where Λ is arbitrary, yield the same physical electromagnetic field. This implies that, although the theory is formulated with these variables, they are redundant to characterize the dynamics of the electromagnetic fields.

The essence of the gauge invariance lies on its *locality*. The transformation we make on A_μ changes point to point, $\Lambda = \Lambda(x)$.

The QFT of Electromagnetism is called Quantum Electrodynamics (QED). QED couples to matter fields, like the Dirac field, by means of covariant derivatives. They are constructions of the forms

$$(\partial_\mu + ieA_\mu)\psi\,, \tag{1.3}$$

where ψ is the Dirac field of the electron and $-e$ is the electron charge.

In the coupled theory the gauge transformation (1.2) must be accompanied of a phase change on ψ

$$\psi' = e^{-ie\Lambda}\psi\,. \tag{1.4}$$

Under a gauge transformation given by Eqs. (1.2) and (1.4) the covariant derivative transforms as ψ does. This transformation is the symmetry principle upon which QED is made, besides the Lorentz symmetry needed by relativity.

The simplest version of QED contains one massless, spin-1 particle, the photon, and one massive, spin-1/2 particle, the electron. The classical Lagrangian density for this theory is (in natural units)

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi\,, \tag{1.5}$$

where γ^μ are constant matrices satisfying the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1.6}$$

which allows to construct spinorial representations of the relativity symmetry group. This is intimately related to the very construction of the Dirac Equation, which can be obtained for a charged particle by varying this Lagrangian with respect to $\overline{\psi}$. $\overline{\psi}$ is the Dirac conjugate to ψ.

The last term in Eq. (1.5) is called a Dirac mass term. The parameter m is the bare electron mass.

In QED the gauge symmetry is intimately related to the fact that the photon is not massive. Indeed a relativistic quantum field cannot be constructed for massless particles of spin-1 unless the theory in formulated in a gauge invariant way. We must say that the rigorous quantization of a gauge theory is far from be a straightforward task.

Yang-Mills theory

Encouraged by the success of QED, Yang and Mills developed a field theory for non-Abelian gauge groups with the hope that it could explain the *nuclear* strong interactions which bounds protons and neutrons in the atomic nucleus.

The Yang-Mills field is a extension of the Maxwell gauge potential A_μ. Besides to be a four-vector, the Yang-Mills field takes values on the algebra of the gauge group. It can be expanded on the basis T^a,

$$A_\mu = A_\mu^a T^a \,. \tag{1.7}$$

As in electrodynamics, in Yang-Mills theory one can introduce charged fields which also transform under the gauge symmetry. They couple to the Yang-Mills gauge field by means of covariant derivatives

$$(\partial_\mu - igA_\mu)\psi \,, \tag{1.8}$$

where g is the Yang-Mills coupling constant.

The gauge transformations of the theory are

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}\partial_\mu U U^{-1} \,, \qquad \psi' = U\psi \,, \tag{1.9}$$

which are local transformations since $U = U(x)$. In addition, there could be other charged fields transforming in a different representation to ψ.

The classical Lagrangian for a Yang-Mills field coupled to a massless field ψ is

$$\mathcal{L} = -\tfrac{1}{4}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + i\overline{\psi}\gamma^\mu(\partial_\mu - igA_\mu)\psi \,. \tag{1.10}$$

Upon quantization, there are particles associated to the Yang-Mills gauge potential as in the case of the photon in QED. The gauge particles are equally massless and with spin 1. However, in non-Abelian gauge theories there are more than one gauge particle because there is one for each dimension of the algebra.

In spite of its completeness, Yang and Mills regarded their theory as a pure mathematical development. This was because no massless, spin-1 particle had been observed in nature besides the photon itself.

Symmetry breaking and the Electroweak Theory

The Fermi's model of the weak interactions governing the β-decays consists of a four-fermion interaction. The scale of the Fermi's model is characterized by the Fermi constant

$$G_F = 1.166\,10^{-5}\ \mathrm{GeV}^{-2} \,. \tag{1.11}$$

The four-fermion nature of the Fermi interaction makes the theory non-renormalizable. However, the model should not be completely wrong since it fits well with the experimental data.

In QED interactions between fermions are mediated by photons. Similarly, the Fermi's model can be cured by substituting it by a Yang-Mills theory in which interactions between fermions are mediated by gauge bosons, rather than the direct four-fermions interactions. However, in a pure Yang-Mills theory the gauge bosons are massless and no massless bosons of this kind had been observed besides the photons.

One can add mass terms for the gauge bosons in the Yang-Mills Lagrangian, breaking explicitly the gauge invariance. However, this kind of theories are also non-renormalizable.

The right way to give masses to the gauge bosons is the mechanism of Spontaneous Symmetry Breaking (SSB), giving masses dynamically at low energies.

In the SSB scenario there exists a bosonic scalar field, ϕ, besides the gauge bosons and the fermions. The scalar transforms under the gauge symmetry in a given representation.

The bosonic scalar feels a potential whose ground level is *degenerated*. The potential is invariant under the gauge symmetry, such that the whole theory is gauge invariant.

At low energies the quantum dynamics of the scalar field is described by perturbations around a classical state, which is given by a minimum of the potential. Therefore the dynamics needs the scalar field to choose a minimum. However, in the SSB scenario the ground states of the potential are such that any of them is *not* gauge invariant.

The simplest case is when the ground states are constant. Then the vacuum is given by an specific constant value

$$< \phi >= \phi_0 \,. \tag{1.12}$$

This value, being physical, is not gauge invariant.

Therefore the choice of a vacuum for the scalar field breaks the gauge symmetry. The theory breaks its symmetries by its own dynamics at low energies.

In the process of symmetry breaking mass terms for the gauge particles are generated. The original theory has a "kinetic" term for the scalar field

$$+[(\partial_\mu - igA_\mu)\phi]^\dagger(\partial^\mu - igA^\mu)\phi \,. \tag{1.13}$$

When the scalar field is described by perturbations around the vacuum,

$$\phi = \phi_0 + \eta \,, \tag{1.14}$$

where η is small, the kinetic term generates a mass term for A_μ,

$$+g^2(A_\mu\phi_0)^\dagger A^\mu\phi_0 \,. \tag{1.15}$$

The gauge group of the Electroweak Theory is $SU(2) \times U(1)$. The potential for the scalar exhibiting SSB is the fourth order polynomial

$$V = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \,, \tag{1.16}$$

whose minima are characterized by

$$\phi^\dagger \phi = \frac{\mu^2}{2\lambda} \,. \tag{1.17}$$

SSB also gives masses to chiral fermions. For this kind of fermions one cannot write a mass term like the one of Eq. (1.5) because it vanishes automatically. There is other possibility, a Majorana mass term, but it breaks the symmetry explicitly

The chiral fermions are coupled to the Higgs scalar in the Yukawa couplings. After the symmetry breaking these couplings give rise to mass terms for the fermions.

Therefore, QED and the weak interaction are treated in an unified frame as the Yang-Mills theory of the gauge group $SU(2) \times U(1)$ with SSB mechanism breaking the gauge group down to $U(1)$.

Quantum Chromodynamics

By doing experiment of inelastic scattering of protons it was discovered that nucleons has internal structure. One of the first ideas to model the structure of nucleons was the model of partons. This model eventually evolved to the one of quarks and gluons.

It was not so easy to realize why the quarks are not observed as free particles but instead strongly bounded forming the nucleons. To solve this problem an extra quantum number was added to the model of quarks, the colour.

Colour is the charge under a fundamental force, the *strong* interaction. Fermions are divided into two classes, quarks and leptons. The former feel the strong interaction whereas the latter not.

The strong interaction is described by a Yang-Mills field with gauge group $SU(3)$. Due to the particle content of the Standard Model, this gauge group exhibits the so called *asymptotic freedom*. The strength of the interaction decreases at high energies and increases at low energies. This behaviour is the opposite to the Electroweak interaction.

The asymptotic freedom is the responsible for the gluon and quark confinement. Since energy scales are the inverses of length scales, in QCD the strength of the interaction grows with distance. If we tried to separate two quarks we would feel an increasing force attracting them.

The particle content of the Standard Model

The Standard Model of fundamental particles is the adding of Electroweak Theory and QCD together with the observed fermions. Therefore the Standard Model is a Yang-Mills theory with gauge group $SU(3)\times SU(2)\times U(1)$ coupled to chiral fermions and one scalar boson.

The scalar boson is called the Higgs scalar. It exhibits a SSB mechanism at low energies breaking the $SU(2)\times U(1)$ sector down to $U(1)$, the QED gauge group.

The gauge group $SU(2)\times U(1)$ has four gauge bosons. After the symmetry breaking, there appears three massive bosons, W^+, W^- and Z^0, which are the carriers of the weak interaction, whereas the photon is the particle corresponding to the unbroken sector of the original symmetry.

The gauge particles of the $SU(3)$ sector are called gluons, they are massless and with spin 1. They have not been observed as asymptotic free sates. This is in agreement with the asymptotic freedom behaviour of QCD.

It has been observed that all fermions belong to one of three families. Each family is like a replica of the others (same quantum numbers), but with different masses.

The fermions are chiral. The left-handed fermions are charged under the $SU(2)$ sector whereas the right-handed do not. There is not right-handed neutrino in the Standard Model.

The fermions are divided into two classes: quarks and leptons. The former feel the strong interaction, they have colour, whereas the latter does not. It is custom to group the fermion into $SU(2)$ doublets. The quarks of the three families, with their funny names, are

$$\begin{pmatrix} \text{up} \\ \text{down} \end{pmatrix}, \qquad \begin{pmatrix} \text{charm} \\ \text{strange} \end{pmatrix}, \qquad \begin{pmatrix} \text{top} \\ \text{bottom} \end{pmatrix}.$$

The last doublet is the most massive one, hence it was the last to be found. The leptons of the three families are

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}, \qquad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \qquad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}.$$

1.1.2 General Relativity

The general relativity principle

Special relativity is restricted to inertial frames. The kind of transformations that go from an observer to other in Special Relativity are the Lorentz transformations. They are a special class of *coordinate transformation* given by

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu} + b^{\mu} , \tag{1.18}$$

where $\Lambda^{\mu}{}_{\nu}$ and b^{μ} are constant. These transformations are the most general ones relating two frames with no acceleration between them. On geometrical grounds, they preserve the line element of *flat* space-time. This is equivalent to demand that the constant matrices $\Lambda^{\mu}{}_{\nu}$ preserve the Minkowskian metric, $\Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$.

A complete theory of kinematics should not be restricted to any special kind of observers. The laws of Physics must be formulated in such a way that they have the same form for all observers. This is the *principle of general relativity.*

The extension to accelerated observers is mathematically equivalent to the invariance under *General Coordinate Transformations* (GCT),

$$x^{\mu\prime} = x^{\mu\prime}(x^{\nu}) . \tag{1.19}$$

Thus, the theory of General Relativity should be invariant under CGT.

GCT has an intrinsic *local* nature. Consider for instance *infinitesimal* GCT. They can be written as

$$x^{\mu\prime} = x^{\mu} + \epsilon^{\mu}(x) , \tag{1.20}$$

where $\epsilon^{\mu}(x)$ are infinitesimal parameters. Any field over space-time transforms in a definite rule under infinitesimal GCT. For instance a scalar field ϕ transforms as

$$\delta\phi = -\epsilon^{\mu}(x)\partial_{\mu}\phi . \tag{1.21}$$

This is evidently a local transformation. General Relativity has naturally GCT as a gauge group.

The inclusion of acceleration leads unavoidably to the inclusion of the gravitation dynamics. Einstein realized this fundamental fact by noting that a body freely falling under the action of a gravitational field does not feel its own weight. Hence the gravitational force can always be locally canceled by an acceleration. This the *equivalence principle.*

Therefore General Relativity cannot be formulated as a pure kinematic theory, it intrinsically includes the gravitational interaction. This feature makes gravitation a very special interaction.

Riemannian geometry

In presence of gravitational fields the space time is not flat. However, one wants still to make calculations since, after all, all the Physics is formulated in terms of variables that depends on time and space. In particular what one specially use is differential calculus, the language of the continuous changes.

Fortunately, by the times of Einstein's studies mathematicians had gained insight into a differential calculus over curved spaces. This is the Differential Geometry and specially the Riemannian geometry. It is the generalization to arbitrary dimension of the measure theory on surfaces.

Differential Geometry allows to formulate the General Relativity in a coordinate-independent way. As we mentioned, this fundamental property lies on the spirit of relativity principles: Physics must be independent of the observer.

The Einstein equations

The premise of General Relativity is that matter/energy curve space-time and at the same time the curvature of space-time determines the movement of matter/energy.

The sources of space-time curvature are represented by a energy-momentum tensor $T_{\mu\nu}$. On the other hand the strength of the gravitational field is the Riemann curvature tensor $R_{\mu\nu\alpha}{}^{\beta}$. These two objects are coupled in the Einstein equations

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}\,, \tag{1.22}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\,, \qquad R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha}\,, \qquad R = g^{\alpha\beta} R_{\alpha\beta}\,. \tag{1.23}$$

To make contact with our intuitive ideas of gravitation, let us see how the Newton's gravitation is recovered from the Einstein equations.

In the limit of weak fields, the vacuum solution to the Einstein equations (1.22) is

$$g_{tt} \sim 1 - \frac{k}{r}\,, \tag{1.24}$$

where k is a constant such that $k << r$. Notice that in this limit the component g_{tt} is basically the Newtonian potential.

On the other hand, General Relativity says that the movement of a point particle of small mass m in the presence of a gravitational is given by the equation

$$\ddot{X}^{\mu} + \Gamma_{\alpha\beta}{}^{\mu} \dot{X}^{\alpha} \dot{X}^{\beta} = 0\,. \tag{1.25}$$

This is true whenever one can neglect the effect of the particle on the gravitational field. On geometrical grounds, this is the equation of *geodesics*. A solution of this equation passing through the points X_1 and X_2 is a *minimal length* curve between these points.

The particle feels the space-time curvature by its coupling to $\Gamma_{\alpha\beta}{}^{\mu}$. We may evaluate the equation (1.25) using the background (1.24). We obtain

$$\ddot{r} = -\frac{c^2 k}{2r^2}\,. \tag{1.26}$$

This is evidently the Newton's Universal Gravitation with its typical inverse square law. k is identified with the source of the gravitation,

$$k = \frac{2G_{\mathrm{N}}M}{c^2}\,. \tag{1.27}$$

This simple exercise helps us to relate the rather abstract idea of a curved space-time with the more familiar notion of force.

Weakness of gravitation

The currently accepted value of the Newton constant is (in natural units)

$$G_{\mathrm{N}} = 0.694\,10^{-38}\ \mathrm{GeV}^{-2}\,. \tag{1.28}$$

With it we can form a mass constant, the Planck mass,

$$M_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G_{\mathrm{N}}}}\,. \tag{1.29}$$

Their value is

$$M_{\mathrm{P}} = 1.22\,10^{19}\ \mathrm{GeV}\,. \tag{1.30}$$

The value of G_{N} is extremely tiny. Indeed G_{N} has been one of the most difficult constants of nature to be measured.

Gravitation and electromagnetism are the only long-range known fundamental interactions. Moreover, gravitation, unlike electromagnetism, is always attractive. For this reason only gravitation is relevant for the dynamics at large (from planetary to cosmological) scales.

1.1.3 Beyond the Standard Model and General Relativity

If the Standard Model and the General Relativity seem to work well, after all they are theoretically well-defined and apparently explain all the experimental data, why do we insist in the search for more fundamental theories? In the next paragraphs we point out some ideas for suggestions of new physics beyond the Standard Model and General Relativity.

In the first plane there are the ideas of unification as philosophical motivation. Secondly we discuss briefly three open questions in high energy Physics: the flavour puzzle, the hierarchy problem and the cosmological constant problem. The latter is particularly interesting because it is a failure arising when one tries to overlap the domain of application of both theories: The cosmological constant problems is essentially a failure of the Standard Model in explaining a cosmological measurement.

The ideas of Unification

We do not feel comfortable with the absence of a quantum description for the gravitational interaction. Gravitation is known to us at large distance scales, but, what is its behaviour at microscopic scales? Does it really exist?

The problem is that the weakness of gravitation makes it very difficult to measure microscopically. Any gravitational effect on particles is exceeded by the other interactions. Nevertheless, we believe that gravitation is a fundamental interaction and hence it is present at all scales.

Microscopic scales are the domain of Quantum Physics. Therefore, we are forced to give a quantum explanation for gravity. Seeing QED, Electroweak and QCD, the first thing one would try to do is a QFT version of the General Relativity. However there is no way to do it consistently because any QFT of General Relativity is non-renormalizable.

The particle carrying the gravitational interaction (the graviton) should be described in a different way. This is one of the most important features of String Theory. It is able to describe the graviton as well as the other particles.

The ideas of unification have been present in high energy physics not only to study quantum gravity, but more simply to extend the unification in Electroweak Theory and Quantum Chromodynamics.

We have already mentioned that Electroweak Theory with SSB unifies electromagnetic and weak interactions. This is not rigorously true since in the gauge group, before the SSB, one direction can be distinguished from the others. Physically this means that there are several coupling constants (several strengths) in the interaction. Mathematically this is due to the presence of an Abelian $U(1)$ factor in the gauge group which is invariant under the rest of the group, it is said that the group is not *semisimple*. Thus Electroweak Theory does not unify the electromagnetic and weak interactions into a *unique* interaction.

Moreover, in the Standard Model QCD and Electroweak interactions are completely unrelated, they are just added. This suggest that there could be a way to unify QCD with Electroweak similar to the case of electromagnetic and weak interactions.

The idea is to find a larger semisimple gauge group with SSB such that it leads to $SU(3) \times SU(2) \times U(1)$ at low energies. These models are called Grand Unification Theories (GUTs).

The flavour puzzle

The Standard Model requires a large number of inputs to which there is no way to predict their values. Therefore they must be measured experimentally. These are for example masses, charges and mixing angles between families.

The large number of free parameters is not the only question in the flavour puzzle. One the most intriguing issues of the SM is why the quark masses spread a wide range of values. The masses of the lightest quarks, u and d are by the order of $\sim$ 1 MeV whereas b is by $\sim$ 1 GeV, a range of one thousand. The quark t is even more heavier.

In the lepton sector the situation is similar, even worse. The recently measured masses for neutrinos are near to 1 eV while the mass of τ is of order 1 GeV. This spans nine orders of magnitude.

Should we accept that the elementary constituents of nature have chosen so different ranges of masses? We rather think that actually they are not the fundamental constituents of nature and their masses are different manifestations of a common structure.

The hierarchy problem

Although there is no precise way to determine the mass of the Higgs boson, its upper bound is near of the electroweak breaking scale, which is < 1 TeV.

Radiative corrections to the Higgs boson mass, computed to one loop, are quadratically divergent,

$$\delta {M_H}^2 \sim \lambda^2 \,, \tag{1.31}$$

where λ is some regularization cutoff.

If we let λ to be as large as the Planck mass, $M_{\rm P} \sim 10^{19}$ GeV, then, in order to maintain the Higgs boson mass below the electroweak breaking scale, we are force to *fine-tune* up to 32 decimal places the bare mass parameter in the Lagrangian.

We think that this is an indication that something new is arising at those scales. Otherwise the renormalization program makes no sense.

This problem in particular gets considerably better in *supersymmetric* extensions of the Standard Model. There the radiative corrections come in form of logarithm of the cutoff. The logarithmic behaviour allows to maintain the growth of the mass much more controlled, no extreme fine-tunning is required.

The cosmological constant

General Relativity includes the possibility of a *vacuum energy*. This means that the gravitational force exists even in absence of matter. In the Einstein-Hilbert Lagrangian this vacuum energy is included by adding a volume term,

$$\frac{1}{16\pi G_{\mathrm{N}}}\int d^4x\sqrt{|g|}(R-2\Lambda)\,, \tag{1.32}$$

where Λ is the cosmological constant.

General Relativity is not able to predict the value of the cosmological constant. Is it as parameter that must be measured. There is a currently measured upper bound on Λ,

$$\Lambda < 10^{-58}\ \mathrm{eV}^2\,. \tag{1.33}$$

Although this bound is really tiny, the cosmological measurement indicates that the cosmological constant is not zero.

One should look at the Standard Model in order to get a prediction for the value of Λ. In the Standard Model there are many effects contributing to the vacuum energy. However, one can get a feeling of the problem by a simple classical consideration related to the electroweak SSB.

The SSB mechanism generates a constant term in the Lagrangian which is the minimum value of the potential (independently of the chosen minimum),

$$V_{\mathrm{min}} = -\frac{\mu^4}{4\lambda}\,. \tag{1.34}$$

This term is completely irrelevant in the Standard Model because it is a shift in the energy that any particle in SM does not feel.

On the other hand, gravity feels any shift in energy because its couples to any kind of energy. Therefore the constant V_{min} is a good candidate for the cosmological constant of General Relativity.

The parameter μ is by the order of the electroweak symmetry breaking and λ is near to one, thus V_{min} yields an estimate of the cosmological constant by the order of

$$\Lambda_{\mathrm{Higgs}} \sim 10^{-12}\ \mathrm{eV}^2\,, \tag{1.35}$$

a failure of almost 50 orders of magnitude respect to the experimental value!

1.2 String Theory

1.2.1 Main ideas

String Theory proposes that the elementary components of nature are not point-like particles but strings. Fundamental strings are characterized by a small length, ℓ_{S}, such that at large enough distance scales we do not see the one-dimensional structure of the string, we instead feel it as a particle.

A stringy structure is much richer than a particle. Strings can vibrate. A spectrum of particles, with various masses and spin, appears as low-energy quantum vibrations of a single string. This is a great advance in simplification.

String theory is formulated with a key ingredient that is supersymmetry. Of course, fermions and bosons are clearly differentiated in nature thus String Theory requires of a supersymmetry-breaking mechanism at low-energies.

The quantum dynamics of supersymmetric strings is only well-defined in ten dimensional space-time. We live in four dimension, hence String Theory should be able to explain what happens with the remaining six dimensions. The standard belief is that extra dimensions are *compactified*. The size of the compactified dimensions is so small that they are only accessible to high energies. Alternatively, it has been postulated that some constituents (the Standard Model) are confined to live in certain dimensions (a brane) while others (gravity) can expand along the extra dimensions. These models are called *braneworlds*.

The possibility of a world with more than four dimensions has stimulated broad theoretical studies about the influence of extra dimensions in the four-dimensional physics. These studies are in the spirit of the Kaluza-Klein ideas. These authors realized that the classical theories of Einstein's gravitation and Maxwell's electromagnetism could be unified into a unique five-dimensional field. Although we know now that classical fields are not appropriated for fundamental theories, the Kaluza-Klein program is one of the most important tools in String Theory.

The most remarkable feature of String Theory is that it contains a quantum dynamics for the gravitational interaction. As we already said, the low-energy spectrum of strings is made of particles of various spin. In the massless sector there is a spin-2 particle. This is the carrier of the gravitational interaction, the graviton. Similarly, String Theory provides a framework for a grand unification of the strong and electroweak interactions.

The fundamental parameter of String Theory is the *Regge slope* α'. It has dimensions of square length (in natural units). The string tension T is the energy per unit length of the string, it is given in terms of α' by

$$T = \frac{1}{2\pi\alpha'} \,. \tag{1.36}$$

In addition, the string length and mass are

$$\ell_{\mathrm{S}} = \sqrt{\alpha'} \,, \qquad m_{\mathrm{S}} = \frac{1}{\sqrt{\alpha'}} \,. \tag{1.37}$$

1.2.2 Basics and origins of String Theory

World-sheet actions

The simplest versions of String Theory are based on the world-sheet formulation. These theories are in some way incomplete since they are quantum relativistic mechanics, i. e. *first quantization*. A complete *quantum string field theory* is still not known. Nevertheless, the world-sheet formulation is appropriated to describe *free* strings.

When the string moves through the space-time it spans a two-dimensional surface, which is called the world-sheet. The theory is described by fields over this surface, which can be parameterized by a time-like coordinate τ and a space-like one σ. In particular the position of the string in the ten-dimensional space is one of such fields. As we said, from the ten-dimensional point of view this a mechanical picture.

The mechanical action of a particle is proportional to the length of its world-line. The obvious analogous for strings is an action proportional to the *area* of its world-sheet Σ,

$$S_{\mathrm{NG}} = -T \int\limits_{\Sigma} \sqrt{-\det\left(g_{\mu\nu}(X)\partial_i X^\mu \partial_j X^\nu\right)} d\tau d\sigma \,, \tag{1.38}$$

where $X^\mu(\tau,\sigma)$, $\mu = 0,\ldots,9$, are the world-sheet fields determining the position of the string in the ten-dimensional space.

$g_{\mu\nu}(X)$ is the ten-dimensional metric. In world-sheet formulation objects belonging to the physical space-time like $g_{\mu\nu}(X)$ play the role of *backgrounds*. They must be given as inputs for each concrete model. Moreover, upon quantization the background fields are interpreted as coupling "constants" susceptible of renormalization. In subsection 1.3.2 we shall see that the metric $g_{\mu\nu}$ cannot be arbitrary.

The object

$$g_{\mu\nu}\partial_\mu X^\mu \partial_\nu X^\nu \tag{1.39}$$

is the pullback of the ten dimensional metric to the world-sheet, although it gives a well-defined pseudo-Riemannian metric only when X^μ are immersions. Therefore

$\sqrt{\det(g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu)}d\tau d\sigma$ is an area element on the world-sheet and clearly the action (1.38) is proportional to the total, induced area. This action for strings is known as the Nambu-Goto action.

The Nambu-Goto action is classically equivalent to the action

$$S_{\mathrm{P}} = -\tfrac{1}{2}T \int_\Sigma d\tau d\sigma \sqrt{-h} h^{ij} \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X)\,, \tag{1.40}$$

where h_{ij} and X^μ are independent variables, h_{ij} being a metric over the world-sheet with no significance in space-time. This action is the Polyakov action. Polyakov and Nambu-Goto actions are classically equivalents because they yield the same equations of motion upon solving the equation for h_{ij} in the Polyakov side, which is an algebraic equation for h_{ij}.

The Polyakov action is advantageous because there is not square root of the X^μ fields. Moreover, in flat background it is quadratic on these fields. In more general backgrounds it is a non-linear σ-model.

The Polyakov action depends on more variables than the Nambu-Goto action. Moreover, it has one further symmetry that is not present in the Nambu-Goto action. Both actions are invariant under GCT on the world-sheet and under isometries of the metric $g_{\mu\nu}$. Polyakov action is also invariant under *Weyl transformations*,

$$h'_{ij} = \Omega(\tau,\sigma)^2 h_{ij}\,. \tag{1.41}$$

This local symmetry plays a crucial role in the quantization of the string. This symmetry is enough to put by gauge fixing the metric in a flat form, $h_{ij} \to \eta_{ij}$ (this is due to the bi-dimensionality of the world-sheet), allowing to put the Polyakov in a extremely simple form.

String can couple to the other background fields. These field are in turn associated to one sector of the string massless modes. They are a space-time scalar ϕ and two-form $B_{\mu\nu}$ fields. The scalar is called the dilaton and $B_{\mu\nu}$ the Kalb-Ramond two form. The coupling of the string to these fields in the world-sheet formulation is

$$+\tfrac{1}{2}T \int_\Sigma d\tau d\sigma \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}(X) - \tfrac{1}{4\pi} \int_\Sigma d\tau d\sigma \sqrt{-h} R(h) \phi(X)\,. \tag{1.42}$$

Notice that in the first term there is not coupling to h_{ij}, hence it is obviously Weyl invariant. The second term is only Weyl invariant under global transformations. Thus this coupling breaks the local Weyl symmetry even at the classical level.

The position operators X^μ are bosonic. In addition one can introduce fermionic fields over the world sheet. The addition of fermionic fields is a physical requirement since otherwise there are not fermionic particles in the string spectrum. Furthermore the vacuum of a purely bosonic string is unstable. By including fermions we arrive at the concept of supersymmetry.

Consider the fields $\psi^\mu(\tau,\sigma)$ which have spin-$\frac{1}{2}$ over the world-sheet and they are vectors on space-time (X^μ are world-sheet scalars and coordinates on space-time). Hence ψ^μ are fermionic degrees of freedom. The action, in a flat background, is

$$S_{\mathrm{P}} = -\tfrac{1}{2}T\int\limits_{\Sigma} d\tau d\sigma\left(\eta^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - i\overline{\psi}^\mu \partial\!\!\!/\, \psi_\mu\right)\,, \tag{1.43}$$

where we have gauged-fixed the auxiliary world-sheet metric using the Weyl symmetry, which is still present with fermions.

Supersymmetry is the condition that the action is invariant under the transformations

$$\delta_\epsilon X^\mu = \overline{\epsilon}\psi^\mu\,, \qquad \delta_\epsilon \psi^\mu = i\,\partial\!\!\!/ X^\mu \epsilon, \tag{1.44}$$

where ϵ is a infinitesimal, fermionic parameter. ϵ is constant and hence the theory has *global* supersymmetry. As can be seen from these variations, supersymmetry mixes bosonic and fermionic degrees of freedom.

A world-sheet action with *local* supersymmetry can also be constructed. It needs the addition of extra fields and the resulting theory can be understood as a bi-dimensional supergravity (although the graviton is not a propagating field in two dimensions).

Supersymmetry can be also introduced in the space-time. One extends the space time by including fermionic coordinates. Thus one ends with a *superspace*. These theories are difficult to quantize in a manifestly Lorentz-covariant way. What is usually done is to quantize them in the light-cone gauge. Both formulations, world-sheet supersymmetry and superspace, are physically equivalent (at least in flat backgrounds).

Anomaly cancellation

The reason for the critical dimension of superstrings (ten dimensions) is anomaly cancellation. This is a rather technical issue, we can say that it is the lacking of a classical symmetry at the quantum level. A symmetry can be present in the classical action, but it does not imply that the *path integral* is invariant. In the case of superstrings propagating through Minkowski space-time, it is precisely the Weyl symmetry that is broken in general at the quantum level. This anomaly is only canceled in ten dimensions.

1.2.3 The various theories of Strings

There are various theories of Superstrings, according to the field content, the number of supersymmetries, the chirality and the gauge groups. There can be open and closed strings and some theories have only closed strings.

The dynamics is completed by specifying boundary conditions. Here there are many possibilities, in the case of open strings one can impose Dirichlet and Neumann conditions at the ends of the string whereas for closed strings the fields must satisfy certain kinds of periodicity conditions. The spectrum of the theory depends strongly on the chosen boundary conditions.

Here we list the five theories of superstrings:

- Type I:

  This is $N = 1$ supersymmetric. Its world-sheet formulation is basically given by the action (1.43).

- Type II, A and B:

  These are $N = 2$ supersymmetric. The relative handedness of the two supersymmetry generators makes the difference between the type A and B. In the type IIA they are of opposite handedness. The spectrum is symmetric in left- and right-handed spinors and hence the theory is *non-chiral*. In the type IIB the supersymmetry generators are of the same handedness and the theory manifest differences in the spectrum between left- and right-handed spinors. This is a *chiral* theory.

- Heterotic, $SO(32)$ and $E_8 \times E_8$:

  These are $N = 1$ supersymmetric. Heterotic strings are a mixture between the bosonic string and the superstring (hence the name). They only contain closed strings. Since left- and right-moving modes on closed strings are independent, the right-moving sector is that of the superstring while the left-movers correspond to the bosonic string. The quantization of the bosonic strings is only well-defined (anomaly-free) is twenty six dimensions. In Heterotic Strings the remaining sixteen ${X^{\mu}}_{\text{left}}$ fields are not interpreted as coordinates of space-time but they are compactified into a *internal* torus. The massless spectrum contains a super Yang-Mills multiplet which gauges either the group $SO(32)$ or $E_8 \times E_8$. In contrast to the purely bosonic string, the vacuum of the heterotic string is stable.

1.2.4 Branes and dualities

One of the most interesting features of String Theory is the physical equivalence between its different formulations. These equivalences are called *dualities*.

T duality was the first duality between string theories to be discovered. It relates the two type II theories and the two heterotic theories and different geometries of the compactified extra dimensions. Consider one spatial dimension being compactified on a circle. One string winded n times around the circle is physically equivalent to other string with momentum proportional to n and with a circle of inverse radius.

S duality relates different coupling regimes. It is a duality between string theories with coupling constant g_S and $1/g_\mathrm{S}$, hence it is a duality between weak/strong couplings. Hence it is a non-perturbative duality between string theories.

Branes play an essential role in the dualities. Among the branes, the D-branes play an essential role because, by definition, they are branes to which open strings can be attached. They are needed to define the T duality in open strings. It turns out that the D-branes are charged under some sector of the massless spectrum of the string.

The quantum dynamics of open strings ending on D-branes yields gauge fields over the D-brane. This has many interesting phenomenological applications.

Dualities elevate the range of branes in String Theory. They are regarded as fundamental elements of the theory as well as the strings.

It has been postulated that there must be a quantum theory covering all the theories of superstrings, that is a mother theory which yields all the formulations of superstrings at different limits and configurations. This hypothetical theory by the moment has a name: M Theory. Most importantly, it is known that the classical effective dynamics of M Theory is $d = 11$ Supergravity. The arising of the eleventh dimension in String Theory was also due to dualities.

1.3 Effective Theories of Strings

1.3.1 Supergravity

Supergravity can be seen as the gauge theory for the Super-Poincaré group. The Poincaré group is the group of the symmetries of flat space-time, they are spatial rotations, rotations between space and time and translations in space and time. The Super-Poincaré group extends these symmetries by including the supersymmetry transformations, which mix bosons and fermions. At first sight it could seem very strange that a symmetry of this kind could be related to space-time symmetries. This is indeed the case, two consecutive supersymmetry transformations give rise to a space-time translation.

On simple grounds we can say that Supergravity is a theory of gravitation coupled to bosonic and fermionic fields in such a way that the theory is supersymmetric. Among these fields there is in particular the supersymmetric partner for the graviton: The gravitino. It is a spin-$\frac{3}{2}$ fermion, also called the Rarita-Schwinger field. The graviton and the gravitino form the *supergravity multiplet*.

As a classical theory, Supergravity can be formulated in several dimensions and with different numbers of supersymmetries. Depending on the dimension there could be extra fields in the Supergravity multiplets.

In the early development Supergravity was also considered as a QFT. For some special cases the supersymmetry makes the theory finite and this a very interesting property. It was though that Supergravity could be a theory of everything. However these theories have anomalies.

Nowadays the importance of Supergravity lies on the fact that it is the low-energy limit of Superstring Theory.

1.3.2 Supergravity as low-energy effective dynamics of strings

As we mentioned, Supergravity is the low-energy limit of String Theory. This means that Supergravity appears in the tree level and $\alpha' \to 0$ limit of strings (at tree level but $\alpha' \neq 0$ there are still stringy modifications to supergravity).

One can see this from three points of view:

- Kinematic arguments:

  The massless spectrum of Strings are in correspondence with supergravity multiplets.

  The five models of superstrings exhibits a massless spectrum of particles which runs for spin-0 (scalars) to spin-2 (tensors) particles, including fer-mions. Each of these modes can be accommodate into supergravity and super Yang-Mills multiplets.

  Although this method does not yield the supergravity action, this is the easiest way to see that the classical limit of strings is Supergravity.

- String amplitudes:

This is the rigorous way. To obtain the classical effective dynamics of a field theory one should compute the amplitudes and then go to a limit where the quantum effects can be neglected. The effective action is the one which reproduce the amplitudes in this limit.

For the case of strings, the classical limit holds at tree level and when $\alpha' \to 0$. This is equivalent to go up to a distance scale where the string length ℓ_{S} can be neglected.

- Weyl invariance:

  As we have mentioned in subsection 1.2.2, in the world-sheet formulations of strings the geometry of the space-time must be given as inputs. Geometrical fields appear in the string action in a similar fashion of coupling constant. Indeed when one computes string amplitudes these fields are subject to renormalization.

  The Weyl symmetry of the theory requires vanishing of the β functions. Since the couplings are just the background fields, the condition $\beta = 0$ yields field equations for the background. It turns out that these equations are just the equation of motion of Supergravity. Actually, this program works for bosonic strings. One extend the conclusion to Supergravity by taking into account the kinematics arguments.

It is illustrative to see the third approach to obtain the string effective action, the Weyl symmetry. We are going to do it for the bosonic string ($d = 26$). As we already say in subsection 1.2.2, the coupling of the string to background field breaks the local Weyl invariance even at the classical level. This is due to the coupling to the dilaton. Most importantly, the Weyl invariance is broken by quantum effects.

To restore the Weyl symmetry one demands the vanishing of the corresponding β functions. To lowest order in α', they are

$$\beta^{g}_{\mu\nu} = \alpha'(R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \tfrac{1}{4}H_\mu{}^{\alpha\beta}H_{\nu\alpha\beta}) + \mathcal{O}(\alpha'^2)\,, \tag{1.45}$$

$$\beta^{B}_{\mu\nu} = \tfrac{1}{2}\alpha' e^{2\phi}\nabla_\alpha(e^{-2\phi}H_{\mu\nu}{}^{\alpha}) + \mathcal{O}(\alpha'^2)\,, \tag{1.46}$$

$$\beta^{\phi} = -\tfrac{1}{2}\alpha'\left[\nabla^2\phi - (\partial\phi)^2 - \tfrac{1}{4}R(g) - \tfrac{1}{48}H^2\right] + \mathcal{O}(\alpha'^2)\,. \tag{1.47}$$

The vanishing of these β functions, up to first order in α', yields a set of equations that are equivalents to the equation of motions of the action

$$S = \frac{g^2}{16\pi G_{\mathrm{N}}{}^{(26)}}\int d^{26}x\sqrt{-g}e^{-2\phi}\left[R - 4(\partial\phi)^2 + \tfrac{1}{2\cdot 3!}H^2\right]\,. \tag{1.48}$$

This a *gravity* action. It is defined in twenty six dimension and in the String frame (observe the scale factor $e^{-2\phi}$ in front of the usual Einstein-Hilbert term). It can be brought to the usual Einstein frame by a conformal rescaling. As we mention, the recovering of the Weyl symmetry yields the equations of motion only for the bosonic sector. For Supergravity, one already knows how to handle with the fermions because the supermultiplets are determined.

1.3.3 Supersymmetric solutions of Supergravity

Unbroken symmetries

Symmetries are present at several levels in Physics. We can find symmetries in mechanical systems as well as in classical and quantum fields. We have seen that the gauge symmetry is the guiding principle for the Standard Model and GCT is for General Relativity. String Theory posses a further symmetry, the supersymmetry.

A theory has a symmetry if it remains (physically) unaltered under a *change* of configurations. Symmetries can also be present a the level of particular configurations, they can posses (some of) the symmetries of the theory. These special configurations remains unaltered under the symmetry transformation of the theory, that is, they are their own images under the transformation.

Symmetric configurations are important for effective theories. For example, in subsection 1.1.1 we mentioned that the SSB mechanism of the Standard Model lies on the fact that the vacuum *breaks* the symmetries of the theory. We may modify the potential (1.16) in such a way the minimum is reached at the zero value of the field (zero is already a critical point of this potential, but it is a local maximum). If we change the sign of the quadratic term, then the minimum of the potential is unique, given by

$$< \phi >= 0\,. \tag{1.49}$$

This configuration is invariant under the symmetry transformation. Roughly speaking, the symmetry rotates the field around the zero point, the zero being obviously its own image under this transformation. Hence we say that this vacuum *preserves* the symmetry. If we make perturbation theory around this vacuum we still have the whole gauge symmetry. Of course, in the Standard Model we choose the potential with the negative sign because we are interested in breaking the symmetries.

In gravity we also have configurations with unbroken symmetries. The gauge symmetry of General Relativity is GCT, which is a infinite-dimensional Lie group. In general a given configuration (metric) will not be invariant under an arbitrary change of coordinates, but it can be invariant under a reduced, finite group of coordinate changes. These are the *isometries* of the metric, their corresponding diffeomorphisms are generated by vectors, called Killing vectors, whose integral lines are directions of symmetry of the geometry (if there is any).

The most relevant example of such unbroken symmetries in gravity is the Minkowski space-time and the Poincaré group. Minkowski space-time is a vacuum solution (without cosmological constant) and the Poincaré group describes the set of coordinate transformations that leave it invariant. Minkowski space-time is an example of *maximally* symmetric space, it is known that the maximum number of symmetries of a particular configurations is $d(d+1)/2$. There are other vacuum solutions that preserves part of the Poincaré group, like the Schwarzschild solution which is spherically symmetric.

Supersymmetry is part of the gauge symmetries of Supergravity. Following the same preceding ideas, there are configurations which preserve some or all of the supersymmetries of the theory. These are called *supersymmetric configurations*. If they are also solutions then they are *supersymmetric solutions*.

By definition, supersymmetric configurations are invariant under a supersymmetry transformation. For infinitesimal, local supersymmetry transformations, supersymmetric configurations satisfy

$$\delta_\epsilon b = \delta_\epsilon f = 0\,, \tag{1.50}$$

for some $\epsilon(x)$, where b and f represent the bosonic and fermionic fields. One is mainly interested in purely bosonic configurations hence one considers all the fermions equal to zero. Thus the condition $\delta_\epsilon b = 0$ is automatically verified. The conditions $\delta_\epsilon f = 0$ are called the Killing spinor equations and solving them for certain supergravity theories is the main objective of this Thesis.

BPS states of Supergravity

Supersymmetric solutions are related to BPS states. These are defined through a bound on the space of parameters (mass and charges) of the configurations.

In the context of non-Abelian Yang-Mills theory, Bogomol'nyi and independently Prasad and Sommerfield (BPS) studied the stability of solitonic configurations. It turns out that the mass and the electric charge are subject to a lower bound (called Bogomol'nyi or BPS bound) that guarantees the stability of the soliton under quantum fluctuations. Among them there is the 't Hooft-Polyakov monopole of the $SU(2)$ gauge theory. The states that saturate the lower bound are called BPS states.

Although those original works were not related to supersymmetry, later on it was realized that any supersymmetric configuration of super Yang-Mills saturates a kind of BPS bound. This can be seen from the very underlying superalgebra. Moreover, the BPS bound is also present in theories with local supersymmetry, Supergravity. This is a way to connect supersymmetry with important physical properties. For example, it has been shown that in any supersymmetric theory the Hamiltonian is positive.

The relation between supersymmetry and BPS states suggests that supersymmetry is present in many physical situations. For example, the *extremal* Reissner-Nordström black hole, although in can be derived in a purely bosonic context, is a BPS state. It saturates the inequality

$$M \geq 2|q|\,, \tag{1.51}$$

where M and q are the mass and the electric charge of the black hole.

BPS states are specially important since they are stable under quantum corrections. Moreover, the BPS bound indicates that the BPS states minimizes the mass. Hence a BPS state cannot decay.

Supersymmetric black holes

Black holes have played a central role in General Relativity. They have been used to study quantum effects in gravity. They are equally important in Supergravity, being the supersymmetric black holes particularly relevant.

A black hole is a region of space-time from which nothing can scape. The boundary of a black hole is called the *event horizon*. Typically one encounters singularities in the region inside the event horizon. In higher dimensions the analogous of black holes are the black p-branes.

In contrast to black holes, there could be solutions of gravity with *naked singularities*, that is singularities which are not surrounded by an event horizon. These configurations are rather unphysical because there is nothing that prevents all matter to be eventually eaten by the singularity. It has been conjectured that this kind of configurations can not be generated dynamically from a regular, initial configuration. This, roughly speaking, is the *cosmic censorship conjecture*.

There is a way to compute the entropy of black holes (also of any asymptotically flat solution) due to Bekenstein and Hawking. It turns out that the entropy is proportional to the horizon area A,

$$S = \frac{1}{4G_{\mathrm{N}}} A\,. \tag{1.52}$$

The area of supersymmetric black holes can be computed in a convenient way and hence their entropy. On the other hand, String Theory is able to give a statistical computation of the entropy of black holes. Since the macroscopic configuration is the result of strings excitations, there could be several strings configurations contributing to the same macroscopic configuration. Then the entropy is proportional to the number of string states contributing to the same macroscopic configuration. If the entropy is statistically computed in this way, then it yields exactly the Bekenstein-Hawking entropy formula (1.52) (to zeroth order in α'). This is a major result of String Theory. It is the only theory able to predict statistically the Bekenstein-Hawking entropy. This can be regarded as a "theoretical laboratory" to test String Theory. Indeed this is the first *quantitative* success of the theory.

* * *

The characterization of supersymmetric solutions of supergravity is fundamental to understand the vacuum structure of String Theory. Also as applications, we may mention three topics of which supersymmetric solutions are useful

- The preserved supersymmetries are fundamental for model building. String compactifications to four dimensions based on supersymmetric vacua are interesting for string phenomenology. As we mentioned in the previous paragraph, supersymmetries preserved by the vacuum are manifested in the effective theory. This is useful to construct supersymmetric extensions of the Standard Model which will give rise to the Standard Model after a supersymmetry breaking.
- Probes for the String/CFT correspondence. To study the conjectured correspondence between String Theory on a bulk and a Conformal Field Theory on the boundary of the bulk the string is placed in supersymmetric backgrounds. This has been achieved in particular in $\mathrm{AdS}_5 \times S^5$, which is a supersymmetric solution.
- Black holes thermodynamics.

We devoted this Thesis to two important theories of supergravity in four and five dimensions. We also use the characterization formalism to gain some insight in the cosmic censorship conjecture.

Let us briefly comment the history of the characterization of supersymmetric solutions in supergravity. The pioneering work was made by Tod [11]. He was encouraged by the discovery of Gibbons and Hull [12] of the presence of a Bogomol'nyi bound on gravitation which is saturated precisely by the supersymmetric solutions of $N = 2$, $d = 4$ Supergravity.

Some years later Tod [5] advanced in the characterization program. He began the analysis for $N = 4$, $d = 4$ Supergravity, which is reconsidered in this Thesis.

Although he completed the characterization of the null case, in the time-like case he found only partial results by imposing certain condition on the Killing spinors, called the "internal rigidity hypothesis". This hypothesis basically consists of the breaking of half of the supersymmetries. It also breaks the $SU(4)$ symmetry of the theory. The same solutions were found independently by Bergshoeff, Kallosh and Ortín [13], who showed that these solutions are generalizations of the Israel-Wilson-Perjés [14,15] solutions of the Einstein-Maxwell theory and include all the known supersymmetric black holes of the theory [16–29].

The topic of supersymmetric solutions of supergravity enjoyed a revival after the work of Ref. [30] showing a new maximally supersymmetric solution of type IIB Supergravity. The solution is analogous to the maximally supersymmetric solution of $d = 11$ Supergravity described by Kowalski and Glikman [31, 32].

Maximally supersymmetric solutions were found quickly in five and six dimensions [33]. Later on it was shown [34] that all maximally supersymmetric solutions in ten and eleven dimensions are:

1. Minkowski, $\mathrm{AdS}_7 \times S^4$, $\mathrm{AdS}_4 \times S^7$ and Kowalski-Glikman for $d = 11$ Supergravity (the latter being called Hpp-waves in Ref. [30]).

2. Minkowski, $\mathrm{AdS}_5 \times S^5$ and Hpp-waves for ten-dimensional supergravities.

We stress that the condition of maximal supersymmetry is quite restrictive hence those configurations are the simplest ones to characterize.

In the work [35] it was shown the characterization of all supersymmetric solutions of pure, ungauged $N = 1$, $d = 5$ Supergravity. This work established the method of the spinor bilinears for classifying the solutions, which is the one we use in this Thesis. Soon after, two of those authors extended the analysis for the gauged theory [36].

After the succees of Ref. [35], Caldarelli and Klemm [37] made the characterization for pure, $U(1)$-gauged $N = 2$, $d = 4$ Supergravity, thus completing the work of Tod. Also it was found in Ref. [38] (see also Ref. [39]) all the supersymmetric solutions of minimal supergravity in six dimensions.

This thesis is organized as follows:

1. In chapter 2 we perform the characterization of supersymmetric solutions of $N = 1$ supergravity in five dimensions coupled to matter vector- and hypermultiplets. The chapter includes the analysis both for the ungauged and gauged cases.

2. Chapter 3 deals with the characterization of supersymmetric solutions of pure $N = 4$ supergravity in four dimensions.

3. In chapter 4 we present in simple grounds how the supersymmetry can be used as criterion to elucidate some physical properties of cosmological solutions. With

this analysis we show the practical importance of knowing the characterization of supersymmetric solutions of supergravity.

4. Finally we point out some conclusions about our results.

2

Supersymmetric solutions of $N = 1$, $d = 5$ Supergravity

In this chapter we will extend further the results obtained in ungauged $N = 1, d = 5$ SUGRA to include, on top of vector multiplets, hypermultiplets. This problem was considered before by Cacciatori, Celi and Zanon in Refs. [1, 2, 40], making progress towards a full solution of the problem which we present here.

Similar works in 4- and 6-dimensional SUGRAs with 8 supercharges ($N = 2, d = 4$ and $N = (1, 0), d = 6$) coupled to vector multiplets and hypermultiplets have been recently published [4, 41]. As the observant reader will see, there is a staggering similarity between the results found in those works and the ones presented here. The reason for this is simply because the hypermultiplets have a very characteristic, and minimal, way of coupling to the rest of the fields, a coupling that is roughly the same in the 3 theories with 8 supercharges, wherefore the resulting structures should be comparable.

2.1 Results

Let us describe the results of this chapter qualitatively: all the supersymmetric solutions can be seen as deformations of supersymmetric solutions with the same electric and magnetic charges but frozen hyperscalars (which is effectively the same as having only vector multiplets), which were classified in Ref. [36]. The effect of defrosting the hyperscalars is an electric and magnetic charge preserving deformation of those solutions; the deformations consist in a deformation of the base space in the timelike case and of the wavefront space in the null case. To be more precise, in the timelike case, the metrics of all the supersymmetric solutions have the general conformastationary form

$$ds^2 = f^2 (dt + \omega)^2 - f^{-1} h_{\underline{mn}} dx^m dx^n \, . \tag{2.1}$$

$h_{\underline{mn}}$ is the time-independent base space metric and when dealing with frozen hypermultiplets, it has to be hyper-Kähler. The metric, with $f = 1$ and $\omega = 0$ and vanishing matter fields is a supersymmetric solution by itself and can be seen as a background which is excited when electric and magnetic charges are turned on. The functions f and ω are essentially determined by the electric and magnetic charges and satisfy covariant differential equations in the base space.

When the hyperscalars are turned on $h_{\underline{mn}}$ is no longer a hyper-Kähler manifold: the form of this metric is dictated by two requirements

1. The hyperscalars $q^X(x)$ are quaternionic maps[1] from the base space to the quaternionic-Kähler target manifold.

2. The anti-selfdual part of the spin connection of the base manifold has to be equal (up to gauge transformations) to the pullback of the $\mathfrak{su}(2)$ connection characterizing the quaternionic-Kähler target manifold.

These two conditions are interwoven but, as we will show in an explicit example, can be solved simultaneously.

Now, the metric, with $f = 1$ and $\omega = 0$, vanishing vector multiplets but unfrozen hyperscalars is a supersymmetric solution by itself and can be seen as a background which is excited when electric and magnetic charges are turned on. The functions f and ω satisfy the same covariant differential equations as before but in the new base space metric.

These solutions generically preserve only 1/8 of the available 8 supersymmetries.

In the null case, the metric is generically of the form

$$ds^2 = 2f du(dv + H du + \omega) - f^{-2} \gamma_{\underline{rs}} dx^r dx^s \, , \tag{2.2}$$

where $r, s = 1, 2, 3$ and all functions are v-independent. The functions f and H and the 1-form ω depend on the electric and magnetic charges and satisfy differential equations in the background of the 3-dimensional wavefront metric $\gamma_{\underline{rs}}$. When the hyperscalars are frozen, this metric is flat; when they are turned on, the 3-dimensional metric is determined by exactly the same two conditions that the base space of supersymmetric solutions of $N = 2, d = 4$ SUGRA coupled to hypermultiplets satisfy, namely

[1]Please see the discussion after Eq. (2.207) for more information about the notion of quaternionic maps.

1. The hyperscalars must satisfy

$$\partial_r q^X \; f_X{}^{iA} \; \sigma^r{}_i{}^j \;=\; 0\,. \tag{2.3}$$

2. The spin connection of the 3-dimensional metric must be equal (up to gauge transformations) to the pull-back of the the $\mathfrak{su}(2)$ connection that characterizes the quaternionic-Kähler target manifold.

This suggests a relation with the 4-dimensional solutions. We thus consider the particular case in which the metric has an additional isometry and is, in particular, u-independent. It is not difficult to see that in general the solutions of the null case describe pp-waves propagating along a string. Solutions which are u-independent can be compactified along the direction in which the wave propagates, *i.e.* along the string and give solutions belonging to the 4-dimensional timelike class, *i.e.* black hole-type solutions.

This set of 5-dimensional solutions and their reductions are presented here for the first time and allow an uplifting of 4-dimensional black-hole-type solutions (with or without hypermultiplets) to $d = 5$ dimensions different from the one considered in Refs. [42–48]. There, 4-dimensional black holes were uplifted to 4-dimensional black holes in a KK monopole background. Here we are dealing with the electric-magnetic dual uplift since the simplest 5-dimensional pp-wave and the Sorkin-Gross-Perry KK monopole [49, 50] are related by dimensional reduction to $d = 4$ dimensions and 4-dimensional electric-magnetic duality, the 4-dimensional solution being the so-called "KK black hole", which in this simple case is singular. This relation is known in the general case under the name of "r-map", whence the r-map will relate these new string-pp-wave upliftings[2] to the known black hole-KK monopole upliftings.

This uplift may be more convenient to understand the black hole solutions from a higher-dimensional point of view since they are direct realizations of the D1-D5-W model. It may shed light on Mathur's conjecture [52,53] on the realization of D1-D5-W microstates as supergravity solutions [54–60]

For the sake of completeness we have also worked out the timelike case with one additional isometry as, with frozen hyperscalars, all of the interesting solutions (supersymmetric rotating black holes and black rings [61]) seem to belong to this class [35, 62, 63]. The base space manifold is now a generalization of the Gibbons-Hawking instanton metric [64]. The Gibbons-Hawking instanton metric is the most general 4-dimensional hyper-Kähler metric with one isometry and can be used as a base space metric $h_{\underline{mn}}$ in absence of hyperscalars. It has the form

[2]A particular case of this kind of uplifting was also observed in Ref. [51], although the 5-dimensional solutions were interpreted as rotating strings.

$$ds^2_{(4)} = H^{-1}(dz + \chi)^2 + H\delta_{\underline{rs}}dx^r dx^s\,, \quad r,s = 1,2,3\,, \tag{2.4}$$

where H is a function harmonic on 3-dimensional Euclidean space.

In presence of unfrozen hyperscalars the metric to be considered is

$$ds^2_{(4)} = H^{-1}(dz + \chi)^2 + H\gamma_{\underline{rs}}dx^r dx^s\,, \quad r,s = 1,2,3\,, \tag{2.5}$$

where the spin connection of the 3-dimensional metric $\gamma_{\underline{rs}}$ has to be equal (up to gauge transformations) to the pullback of the $\mathfrak{su}(2)$ connection of the hyperscalar manifold.

2.2 Matter-coupled, ungauged $N = 1, d = 5$ supergravity

In this section we describe briefly the supergravity theories we will be working with: $N = 1, d = 5$ (minimal) ungauged supergravity coupled to n_v vector multiplets and n_h hypermultiplets[3].

The supergravity multiplet consists of the graviton $e^a{}_\mu$, the graviphoton A_μ and the gravitino ψ^i_μ. The gravitino and the rest of spinors in the theory are pairs of symplectic-Majorana spinors $i = 1, 2$ as explained in Appendix B.2.1.

Each of the n_v vector multiplets, labeled by $x = 1, \cdots, n_v$ consists of one real vector field A^x_μ, a real scalar ϕ^x and a gaugino λ^{xi}. The scalars ϕ^x, parametrize a Riemannian manifold which we call "the scalar manifold". The full theory is formally invariant under an $SO(n_v + 1)$ symmetry that mixes the matter vectors $A^x{}_\mu$ with the supergravity vector $A_\mu \equiv A^0{}_\mu$ and so it is convenient to treat all the vector fields on the same footing denoting them by $A^I{}_\mu$ $I = 0, \cdots, n_v$. The symmetry that rotates the vectors acts on the scalars as well and, to make it manifest one defines $n_v + 1$ functions of the physical scalars $h^I(\phi)$. These functions satisfy the constraint

$$C_{IJK}h^I h^J h^K = 1\,, \tag{2.6}$$

where C_{IJK} is a fully symmetric real constant tensor which characterizes completely the couplings in the vectorial sector. In particular it determines the metric of the scalar manifold $g_{xy}(\phi)$ on the target of ϕ^x, the couplings between scalars and vector

[3]We follow essentially the notation and conventions of Ref. [65] with some minor changes to adapt them to those in Refs. [66, 67]. The original references on matter-coupled $N = 1, d = 5$ SUGRA are [68] and [69]. The origin of these theories from compactifications of 11-dimensional supergravity on Calabi-Yau 3-folds was studied in Ref. [70].

fields $a_{IJ}(\phi)$ and the coupling constants of the vector field Chern-Simons terms. The relations between these fields are given in the Appendix D.1.

Each of the n_h hypermultiplets consists of four real scalar-fields (*hyperscalars*) q^X, $X = 1, \cdots, 4n_h$ and two spinor fields (*hyperinos*) ζ^A, $A = 1, \ldots, 2n_h$. The index i associated to the symplectic-Majorana condition is embedded into the index A. The hyperscalars q^X parametrize a quaternionic-Kähler manifold, described in Appendix D.2, that we will refer to as the *hypervariety*. In particular we observe that the connection of quaternionic-Kähler manifolds can be decomposed in an $\mathfrak{sp}(1) \simeq \mathfrak{su}(2)$ and an $\mathfrak{sp}(n_h)$ component whose pullback to spacetime will act on objects with index i and A, respectively.

The bosonic part of the action is

$$\begin{aligned} S &= \int d^5x\sqrt{g}\,\Big\{R + \tfrac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y + \tfrac{1}{2}g_{XY}\partial_\mu q^X\partial^\mu q^Y \\ &\qquad -\tfrac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu} + \tfrac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha\Big\}\,. \end{aligned} \tag{2.7}$$

Observe that the hyperscalars do not couple to any of the fields in the vector multiplets and couple to the supergravity multiplet only through the metric. This is similar to what happens in $N = 2, d = 4$ theories and will have similar consequences.

We use the following notation for the equations of motion

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{g}}\frac{\delta S}{\delta e^a{}_\mu}\,, \quad \mathcal{E}_x \equiv -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta\phi^x}\,, \quad \mathcal{E}_X \equiv -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta q^X}\,, \quad \mathcal{E}_I{}^\mu \equiv \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A^I{}_\mu}\,, \tag{2.8}$$

which are given by

$$\begin{aligned} \mathcal{E}_{\mu\nu} &= G_{\mu\nu} - \tfrac{1}{2}a_{IJ}\left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\right) \\ &\qquad + \tfrac{1}{2}g_{xy}\left(\partial_\mu\phi^x\partial_\nu\phi^y - \tfrac{1}{2}g_{\mu\nu}\partial_\rho\phi^x\partial^\rho\phi^y\right) \\ &\qquad + \tfrac{1}{2}g_{XY}\left(\partial_\mu q^X\partial_\nu q^Y - \tfrac{1}{2}g_{\mu\nu}\partial_\rho q^X\partial^\rho q^Y\right)\,, \end{aligned} \tag{2.9}$$

$$g^{xy}\mathcal{E}_y = \mathfrak{D}_\mu\partial^\mu\phi^x + \tfrac{1}{4}g^{xy}\partial_y a_{IJ}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\,, \tag{2.10}$$

$$g^{XY}\mathcal{E}_Y = \mathfrak{D}_\mu\partial^\mu q^X\,, \tag{2.11}$$

$$\mathcal{E}_I{}^{\mu} \;=\; \nabla_\nu(a_{IJ}F^{J\,\nu\mu}) + \tfrac{1}{4\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}F^J{}_{\nu\rho}F^J{}_{\sigma\alpha}\,. \tag{2.12}$$

To these definitions we add the following notation for the Bianchi identities of the vector fields:

$$\mathcal{B}^I{}_{\mu\nu\rho} \equiv 3\nabla_{[\mu}F^I{}_{\nu\rho]}\,. \tag{2.13}$$

In these equations $\mathfrak{D}_\mu$ is the covariant derivative in the spacetime and in the corresponding scalar manifold. Then, Eq. (2.11) states that q is a harmonic map from spacetime to the hypervariety.

The supersymmetry transformation rules for the fermionic fields, evaluated on vanishing fermions, are

$$\delta_\epsilon\psi^i_\mu \;=\; \mathsf{D}_\mu\epsilon^i - \tfrac{1}{8\sqrt{3}}h_I F^{I\,\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4g_{\mu\alpha}\gamma_\beta\right)\epsilon^i\,, \tag{2.14}$$

$$\delta_\epsilon\lambda^{ix} \;=\; \tfrac{1}{2}\left(\not{\partial}\phi^x - \tfrac{1}{2}h^x_I\;\not{F}^I\right)\epsilon^i\,, \tag{2.15}$$

$$\delta_\epsilon\zeta^A \;=\; \tfrac{1}{2}f_X{}^{iA}\;\not{\partial}q^X\epsilon_i\,, \tag{2.16}$$

where D_μ is the Lorentz- and $SU(2)$-covariant derivative

$$\mathsf{D}_\mu\epsilon^i \equiv \nabla_\mu\epsilon^i + \epsilon^j\mathsf{A}_j{}^i{}_\mu\,, \tag{2.17}$$

and the $\mathfrak{su}(2)$ connection is the pullback of the $\mathfrak{su}(2)$ connection of the hypervariety:

$$\mathsf{A}^r{}_\mu \;\equiv\; \partial_\mu q^X\;\omega_X{}^r\,, \qquad \mathsf{A}_j{}^i \;=\; i\mathsf{A}^r\;\sigma^r{}_j{}^i\,. \tag{2.18}$$

Observe that the hyperscalars only appear in the gravitino's and gauginos' supersymmetry transformation rules precisely through the $\mathfrak{su}(2)$ connection.

Finally, the supersymmetry transformation rules of the bosonic fields are

$$\delta_\epsilon e^a{}_\mu \;=\; \tfrac{i}{2}\bar{\epsilon}_i\gamma^a\psi^i_\mu\,, \tag{2.19}$$

$$\delta_\epsilon A^I{}_\mu \;=\; -\tfrac{i\sqrt{3}}{2}h^I\bar{\epsilon}_i\psi^i_\mu + \tfrac{i}{2}h^I_x\bar{\epsilon}_i\gamma_\mu\lambda^{x\,i}\,, \tag{2.20}$$

$$\delta_\epsilon\phi^x \;=\; \tfrac{i}{2}\bar{\epsilon}_i\lambda^{x\,i}\,, \tag{2.21}$$

$$\delta_\epsilon q^X = -i f_{iA}{}^X \bar{\epsilon}^i \zeta^A \,. \tag{2.22}$$

2.3 KSIs and integrability conditions

The bosons' supersymmetry transformation rules lead to the following KSIs [6, 71] associated to the gravitino, gauginos and hyperinos *resp.*:

$$\left(\mathcal{E}_\mu{}^\nu \gamma_\nu + \tfrac{\sqrt{3}}{2} h^I \mathcal{E}_{I\,\mu}\right) \epsilon^i = 0\,, \tag{2.23}$$

$$\left(\mathcal{E}_x - h^I_x \not{\mathcal{E}}_I\right) \epsilon^i = 0\,, \tag{2.24}$$

$$f_{iA}{}^X \mathcal{E}_X \epsilon^i = 0\,. \tag{2.25}$$

It is an implicit assumption, used to derive the KSIs, that the Bianchi identities are satisfied. This affects, in particular, the first two KSIs, where the vector field equations appears. It is, therefore, useful to derive them from the integrability conditions of the KSEs, even if the derivation requires much more work, because in this case, contrary to what happens in $N = 2, d = 4$ theories [3], there is no electric-magnetic symmetry indicating in what combination the Bianchi identities should accompany the Maxwell equations.

The integrability condition of the KSE associated to the gravitino supersymmetry transformation gives

$$\begin{aligned} 4\gamma^\nu \mathsf{D}_{[\mu}\delta_\epsilon \psi^i_{\nu]} &= \Big\{\left(\mathcal{E}_\mu{}^\sigma - \tfrac{1}{3} g_\mu{}^\sigma\, \mathcal{E}_\rho{}^\rho\right)\gamma_\sigma \\ &\quad + \tfrac{1}{4\sqrt{3}} h^I \left[\gamma_\mu \left(\not{\mathcal{E}}_I + \tfrac{1}{6} a_{IJ}\, \not{\mathcal{B}}^J\right) + 3\left(\not{\mathcal{E}}_I + \tfrac{1}{6} a_{IJ}\, \not{\mathcal{B}}^J\right)\gamma_\mu\right]\Big\}\, \epsilon^i = 0\,. \end{aligned} \tag{2.26}$$

To obtain this equation we need to use Eqs. (D.30)-(D.32), with $\nu = -1$ as to ensure the correct normalization of the hyperscalars' energy-momentum tensor. It is a well-known result that manifolds with the opposite sign of ν cannot be coupled to supergravity and here we are just recovering this result.

Acting with γ^μ from the left, we get

$$\left[\mathcal{E}_{\rho}{}^{\rho} + \tfrac{\sqrt{3}}{2}h^{I}(\not{\mathcal{E}}_{I} - \tfrac{1}{3}a_{IJ}\,\not{\mathcal{B}}^{J})\right]\epsilon^{i} = 0\,, \tag{2.27}$$

which can be used to eliminate $\mathcal{E}_{\rho}{}^{\rho}$ from the integrability equation:

$$\left[\left(\mathcal{E}_{\mu}{}^{\sigma} + \tfrac{\sqrt{3}}{2}h_{I}{}^{\star}\mathcal{B}^{I}{}_{\mu}{}^{\sigma}\right)\gamma_{\sigma} + \tfrac{\sqrt{3}}{2}h^{I}\mathcal{E}_{I\,\mu}\right]\epsilon^{i} = 0\,. \tag{2.28}$$

On the other hand, from the gauginos' supersymmetry transformation rule we get

$$2\,\not{\mathfrak{D}}\delta_{\epsilon}\lambda^{ix} = \left[\mathcal{E}_{x} - h^{I}_{x}\left(\not{\mathcal{E}}_{I} + \tfrac{1}{6}a_{IJ}\,\not{\mathcal{B}}^{J}\right)\right]\epsilon^{i} = 0\,. \tag{2.29}$$

Eqs. (2.28) and (2.29) are the modifications to the two KSIs Eq. (2.23) and Eq. (2.24) that we were seeking for.

Let us now obtain tensorial equations form the spinorial KSIs: acting with $i\bar{\epsilon}_{i}\gamma_{\rho}$ from the left on Eq. (2.28) and taking into account the properties of the spinor bilinears discussed in Appendix B.2.2, we get

$$f\left(\mathcal{E}_{\mu\rho} + \tfrac{\sqrt{3}}{2}h_{I}{}^{\star}\mathcal{B}^{I}{}_{\mu\rho}\right) + \tfrac{\sqrt{3}}{2}h^{I}\mathcal{E}_{I\,\mu}V_{\rho} = 0\,, \tag{2.30}$$

whose symmetric and antisymmetric parts give independent equations.

Doing the same on Eqs. (2.29) and (2.25), we get

$$\mathcal{E}_{x}V^{\rho} - fh^{I}_{x}\mathcal{E}_{I}{}^{\rho} = 0\,, \tag{2.31}$$

$$\mathcal{E}_{X}V^{\rho} = 0\,. \tag{2.32}$$

Finally, acting with $i\bar{\epsilon}_{i}$ on Eqs. (2.28), (2.29) and (2.25) from the left we get respectively

$$\left(\mathcal{E}_{\mu\rho} + \tfrac{\sqrt{3}}{2}h_{I}{}^{\star}\mathcal{B}^{I}{}_{\mu\rho}\right)V^{\rho} + \tfrac{\sqrt{3}}{2}fh^{I}\mathcal{E}_{I\,\mu} = 0\,, \tag{2.33}$$

$$f\mathcal{E}_{x} - h^{I}_{x}\mathcal{E}_{I\,\rho}V^{\rho} = 0\,, \tag{2.34}$$

$$\mathcal{E}_{X}f = 0\,. \tag{2.35}$$

which can be obtained from Eqs. (2.30)-(2.32) only in the timelike $f \neq 0$ case.

Summarizing, in the timelike case, defining the unimodular timelike vector $v^{\mu} \equiv V^{\mu}/f$, we have

$$\mathcal{E}^{\mu\nu} = -\tfrac{\sqrt{3}}{2} h^I \mathcal{E}_I{}^{(\mu} v^{\nu)}\,, \tag{2.36}$$

$$h_I {}^\star\mathcal{B}^{I\,\mu\nu} = -h^I \mathcal{E}_I{}^{[\mu} v^{\nu]}\,, \tag{2.37}$$

$$\mathcal{E}_x = h^I_x \mathcal{E}_{I\rho} v^\rho\,, \tag{2.38}$$

$$\mathcal{E}_X = 0\,, \tag{2.39}$$

which imply that all the supersymmetric configurations automatically solve the equation of motion of the hyperscalars and that, if the Maxwell equations are satisfied, then the Einstein and scalar equations and the projections $h_I \mathcal{B}^I$ of the Bianchi identities are also satisfied. Therefore, in the timelike case, the necessary and sufficient condition for a supersymmetric configuration to also be a solution of the theory, is that it must solve the Maxwell equations and the Bianchi identities. Observe that, contrary to the 4-dimensional cases in which only one component of the Maxwell equations and Bianchi identities (the time component) need to be checked because the rest are automatically satisfied, in this 5-dimensional case we need to check all the components of the Maxwell equations and of the Bianchi identities.

In the null ($f = 0$) case, we get, renaming V^μ as l^μ

$$\mathcal{E}_{\mu\rho} l^\rho = -\tfrac{\sqrt{3}}{2} h_I {}^\star\mathcal{B}^I{}_{\mu\rho} l^\rho\,, \tag{2.40}$$

$$h^I \mathcal{E}_{I\,\mu} = 0\,, \tag{2.41}$$

$$h^I_x \mathcal{E}_{I\,\rho} l^\rho = 0\,, \tag{2.42}$$

$$\mathcal{E}_x = 0\,, \tag{2.43}$$

$$\mathcal{E}_X = 0\,, \tag{2.44}$$

which imply that the scalar and hyperscalars equations are automatically satisfied and so are certain projections of the Maxwell and Einstein equations.

2.4 Supersymmetric configurations and solutions

2.4.1 General setup and first results

In this section we will follow the procedure of Ref. [35] to obtain supersymmetric configurations of supergravity, which consists in deriving equations for all the bilinears that can be constructed from the Killing spinors. These equations contain the lion's part of the information contained in the KSEs and can be used to constrain the form of the bosonic fields. These constraints are necessary conditions for the configurations to be supersymmetric and subsequently one has to prove that they are also sufficient (or find the missing conditions, as will happen in the null case). Finally one has to impose the equations of motion on the supersymmetric configurations in order to have classical supersymmetric solutions. The KSIs, derived in the previous section, simplify this task since only a small number of equations of motion are independent for supersymmetric configurations.

As we remarked in section 2.2, the hyperscalars appear only implicitly in the gravitino and gauginos supersymmetry transformations through the pullback of the $\mathfrak{su}(2)$ connection. The equations we are going to obtain for the fields in the supergravity and vector multiplets are, therefore, formally identical to the case without hypermultiplets considered in Ref. [72], but containing implicitly the $\mathfrak{su}(2)$ connection and its consequences. This is similar to what happens in the coupling of $N=2, d=4$ theories to hypermultiplets considered only recently in Ref. [4]

Our goal is to find all the field configurations for which the KSEs

$$\left\{\mathsf{D}_\mu - \tfrac{1}{8\sqrt{3}}h_I F^{I\,\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4g_{\mu\alpha}\gamma_\beta\right)\right\}\epsilon^i = 0\,, \tag{2.45}$$

$$\left(\not\partial\phi^x - \tfrac{1}{2}h_I^x\ \not F^I\right)\epsilon^i = 0\,, \tag{2.46}$$

$$f_X{}^{iA}\,\not\partial q^X\epsilon_i = 0\,, \tag{2.47}$$

admit at least one solution ϵ^i. We are going to assume its existence and we are going to derive necessary conditions for this to happen. These conditions will arise as consistency conditions of the equations satisfied by the tensors that can be constructed as bilinears of the Killing spinor whose existence was assumed from the onset.

As explained in Appendix (B.2.2), the tensor-bilinears that can be constructed from a symplectic-Majorana spinor are a scalar f, a vector V and three 2-forms Φ^r. f and V are $SU(2)$-singlets whereas the Φs form an $SU(2)$-triplet.

The fact that the Killing spinor satisfies Eq. (2.45) leads to the following differential equations for the bilinears:

$$df = \tfrac{1}{\sqrt{3}} h_I i_V F^I \,, \tag{2.48}$$

$$\nabla_{(\mu} V_{\nu)} = 0\,, \tag{2.49}$$

$$dV = -\tfrac{2}{\sqrt{3}} f h_I F^I - \tfrac{1}{\sqrt{3}} h_I {}^\star \left(F^I \wedge V\right)\,, \tag{2.50}$$

$$\mathsf{D}_\alpha \Phi^r{}_{\beta\gamma} = -\tfrac{1}{\sqrt{3}} h_I F^{I\,\rho\sigma} \left(g_{\rho[\beta}{}^\star\Phi^r{}_{\gamma]\alpha\sigma} - g_{\rho\alpha}{}^\star\Phi^r{}_{\beta\gamma\sigma} - \tfrac{1}{2} g_{\alpha[\beta}{}^\star\Phi^r{}_{\gamma]\rho\sigma}\right)\,, \tag{2.51}$$

where

$$\mathsf{D}_\alpha \Phi^r{}_{\beta\gamma} = \nabla_\alpha \Phi^r{}_{\beta\gamma} + 2\varepsilon^{rst}\mathsf{A}^s{}_\alpha \Phi^t{}_{\beta\gamma}\,. \tag{2.52}$$

These equations are formally identical to those obtained in Ref. [72] but now the covariant derivative that acts on the triplet of 2-forms is an $SU(2)$-covariant derivative.

Eqs. (2.46) and (2.47) lead to algebraic equations for the tensor bilinears: contracting Eq. (2.46) with $i\bar{\epsilon}_i$ and $\sigma^r{}_i{}^j\bar{\epsilon}_j$ we get

$$\pounds_V \phi^x = 0\,, \tag{2.53}$$

$$h^x_I F^I_{\alpha\beta}\Phi^{r\,\alpha\beta} = 0\,, \tag{2.54}$$

and the contraction of Eq. (2.47) with $i\bar{\epsilon}_k$ yields

$$\pounds_V q^X = 0\,. \tag{2.55}$$

Contracting now Eq. (2.46) with $i\bar{\epsilon}_i\gamma^\mu$ and $\sigma^r{}_i{}^j\bar{\epsilon}_j\gamma^\mu$ we get

$$f d\phi^x = -h^x_I i_V F^I\,, \tag{2.56}$$

$$0 = \Phi^r{}_{\mu\nu}\partial^\nu\phi^x + \tfrac{1}{4}\varepsilon_{\mu\nu\alpha\beta\gamma} h^x_I F^{I\,\nu\alpha}\Phi^{r\,\beta\gamma}\,, \tag{2.57}$$

and, finally, operating on Eq. (2.47) with $\bar{\epsilon}_k\gamma^\mu$

$$f\partial_\mu q^X + \Phi^r{}_\mu{}^\nu\partial_\nu q^Y J^r{}_Y{}^X = 0\,, \tag{2.58}$$

where we have identified the complex structures of the target quaternionic-Kähler manifold,

$$J^r{}_Y{}^X = f_Y{}^{iA}\mathsf{J}^r{}_{iA}{}^{jB}f_{jB}{}^X\,. \tag{2.59}$$

Eq. (2.185) says that V is an isometry of the space-time metric. The differential equation of Φ^r (2.187) implies

$$d\Phi^r + 2\varepsilon^{rst}\mathsf{A}^s\wedge\Phi^t = 0\,, \tag{2.60}$$

i.e. the three 2-forms are covariantly closed respect to the induced $\mathfrak{su}(2)$ connection.

In order to make further progress, it is necessary to separate the timelike ($f\neq 0$) and null ($f=0$) cases.

2.4.2 The timelike case

The equations for the bilinears

In this case the Killing vector V is a timelike, $V^2 = f^2 > 0$. We introduce an adapted time coordinate t: $V=\partial_t$. With this choice of coordinates the metric can be decomposed in the following way

$$ds^2 = f^2\left(dt+\omega\right)^2 - f^{-1}h_{\underline{mn}}dx^m dx^n\,, \tag{2.61}$$

where ω is a time-independent 1-form and $h_{\underline{mn}}$ is a time-independent Riemannian four-dimensional metric.[4] Eqs. (2.48),(2.190) and (2.197) imply that with our choice of coordinates the scalars f, ϕ^x and q^X are time-independent.

Following Ref. [35] we define the following decomposition

$$fd\omega = G^+ + G^-\,, \tag{2.62}$$

where G^+ and G^- are the selfdual and anti-selfdual parts respect to the metric h.

The Fierz identity Eq. (B.99) indicates that the Φ^rs have no time components and the Fierz identity Eq. (B.100) implies that they are anti-selfdual respect to the spatial metric h. Moreover, the identity Eq. (B.101) becomes

$$\Phi^r{}_m{}^n\Phi^s{}_n{}^p = -\delta^{rs}\delta_m{}^p + \varepsilon^{rst}\Phi^t{}_m{}^p\,, \tag{2.63}$$

[4] Appendix C.3 contains a Vielbein basis and the non-vanishing components of the connection and Ricci tensor in that basis.

where all operations on the spatial indices refers to the spatial metric h. This is the algebra of the imaginary unit quaternions, whence we may conclude that the spatial manifold is endowed with an *almost* quaternionic structure.

The next step is to obtain the form of the supersymmetric vector fields from Eqs. (2.48), (2.186), (2.191) and (2.193): these equations contain no explicit contributions from the hyperscalars and, therefore lead to the same form of the vector fields found in Ref. [72], namely

$$F^I = -\sqrt{3}\{d\left[fh^I(dt+\omega)\right] + \Theta^I\}\,, \tag{2.64}$$

where the Θ^Is are spatial selfdual 2-forms and

$$G^+ = -\tfrac{3}{2}h_I\Theta^I\,. \tag{2.65}$$

From (2.187) information about the derivatives of the two-forms Φ^r can be extracted using the above expression for F^I: first, by introducing the spin connection of the metric given in Appendix C.3 we may obtain the spatial components of the five-dimensional covariant derivative,

$$\nabla^{(5)}_m\Phi^r{}_{nq} = f^{3/2}\nabla_m\Phi_{nq} - \tfrac{2}{3}\left(\delta_{m[n}\partial_{p]}f^{3/2}\Phi^r{}_{pq} - \delta_{m[q}\partial_{p]}f^{3/2}\Phi^r{}_{pn} - \partial_m f^{3/2}\Phi^r{}_{nq}\right)\,, \tag{2.66}$$

where ∇_m is the covariant derivative of the four-dimensional spatial metric. On the right hand side of this expression all of the flat indices refers to the Vielbein $v_m{}^{\underline{i}}$. On the other hand, the spatial components of the equation (2.187) are

$$\nabla^{(5)}_m\Phi^r{}_{nq} + 2f^{3/2}\varepsilon^{rst}\mathsf{A}^s{}_m\Phi^t{}_{nq} = -\tfrac{1}{\sqrt{3}}fh_IF^{I\,p0}\left(\delta_{p[n}\Phi^r{}_{q]m} - \delta_{pm}\Phi^r{}_{nq} - \delta_{m[n}\Phi^r{}_{q]p}\right) \tag{2.67}$$

where we have used the fact that Φ^r are spatial, anti-selfdual 2-forms. Now from Eq. (2.201) we read

$$h_IF^{I\,p0} = \sqrt{3}f^{-1/2}\partial_p f \tag{2.68}$$

and by comparing Eqs. (2.66) and (2.67) we find that the 2-forms Φ^r are $SU(2)$- and Lorentz-covariantly constant over the 4-dimensional spatial manifold:

$$\nabla_m\Phi^r{}_{np} + 2\varepsilon^{rst}\mathsf{A}^s{}_m\Phi^t{}_{np} \;=\; \partial_m\Phi^r{}_{np} - 2\xi_{m[n|}{}^q\Phi^r{}_{q|p]} + 2\varepsilon^{rst}\mathsf{A}^s{}_m\Phi^t{}_{np} \;=\; 0\,, \tag{2.69}$$

Here ξ is the standard spin connection of the 4-dimensional spatial manifold.

Had the base space not been 4-dimensional, the conclusion would have been that we are dealing with a quaternionic-Kähler manifold. But in four dimensions the above equation, taken at face value, is rather void: given a Vierbein we can construct a kosher quaternionic structure by inducing the one from $\mathbb{R}^4$ and then the unique A solving Eq. (2.210), is given by

$$\mathsf{A}^r_m \;=\; \tfrac{1}{16}\,\varepsilon^{rts}\;\Phi^{t\,n}_{p}\;\nabla_m\Phi^{s\,p}_{n}\;. \tag{2.70}$$

In the case at hand, however, said arbitrariness is nothing but an illusion since the connection A is the one induced from an $\mathfrak{sp}(1)$ connection on a quaternionic-Kähler manifold and is therefore not to be chosen but to be deduced. At this point one can then already appreciate the interwoven nature of the problem: Since the quaternionic structure on the 4-dimensional space is basically known, Eq. (2.210) determines, part of, the spin connection in terms of the pull-back of an $\mathfrak{sp}(1)$ connection. This pull-back, however, is defined by means of a harmonic map satisfying Eq. (2.195), which presupposes knowing the Vierbein, and hence also the spin connection.

A ‘trivial’ solution to the requirement that the hyperscalars form a harmonic map satisfying Eq. (2.195), is to take them to be constant: Eq. (2.210) then states that Φ defines a covariantly constant hypercomplex structure, so that the 4-dimensional manifold has to be hyper-Kähler, and we recover the results of [35, 72]. As is well-known the holonomy of a 4-dimensional hyper-Kähler space is $\mathfrak{su}(2)\subset\mathfrak{so}(4)$, and in a suitable frame the spin connection can be taken to be selfdual. The technical reason why the spin connection can be taken to be selfdual lies in the fact that the Φs are anti-selfdual and that the split into anti- and selfdual components corresponds to the Lie algebraic split $\mathfrak{so}(4)\;\cong\;\mathfrak{su}(2)_+\oplus\mathfrak{su}(2)_-$; if we then take the Φs to be induced from the ones on $\mathbb{R}^4$, called J, and denote the projection of the spin connection onto $\mathfrak{su}(2)_\pm$ by $\xi^\pm$, then Eq (2.210) can be expressed as $[\xi^-_m,\mathsf{J}^r]=0$, which immediately implies $\xi^-=0$.

In the general case there will still be no constraint on ξ^+, but we can solve equation (2.210) to give

$$\xi^-_{m\;n}{}^{q} \;=\; -\vec{\mathsf{A}}_m\;\cdot\;\vec{\mathsf{J}}_n{}^{q}\;, \tag{2.71}$$

where as above, we made use of the quaternionic structure induced from flat space.

In the above we were able to match things up without much ado, since the relevant $\mathfrak{su}(2)$s both acted in the vector representation. When considering the Killing spinor equation, however, the representations do not add up that nicely, and one finds that a necessary condition for having unbroken supersymmetry is that the generators of $\mathfrak{su}(2)$ and $\mathfrak{su}(2)_-$ should have identical actions on the Killing spinors, *i.e.*

$$\epsilon^j \; i\sigma^r{}_j{}^i \;=\; \tfrac{1}{4}\, \mathsf{J}^r{}_{mn}\gamma^{mn}\, \epsilon^i\,, \tag{2.72}$$

and these conditions will appear as projectors $\Pi^{r\,+}{}_i{}^j$ acting on the Killing spinors, where

$$\Pi^{r\,\pm}{}_i{}^j \;=\; \tfrac{1}{2}\left[\,\delta \;\pm\; \tfrac{i}{4}\, \not{\mathsf{J}}^{(r)}\sigma^{(r)}\right]_i{}^j\,. \tag{2.73}$$

In principle we only need to impose one such constraint for each non-trivial component A^r.

The last constraint on the bosonic fields comes from Eq. (2.195). In the timelike case this equation is purely spatial and in 4-dimensional notation reads

$$\partial_m q^X \;=\; \Phi^r{}_m{}^n\, \partial_n q^Y\; J^r{}_Y{}^X\,. \tag{2.74}$$

This condition implies that q is what Ref. [73] calls a *quaternionic map.* In said reference it is shown that a quaternionic map between hyper-Kähler manifolds implies that the map is harmonic, *i.e.* it solves

$$\mathfrak{D}_\mu\partial^\mu\; q^X = 0\,. \tag{2.75}$$

Here, however, we are not dealing with maps between hyper-Kähler manifolds, yet the KSIs state that q is automatically harmonic. The question then is: Apart from being quaternionic, what properties must q satisfy in order to be harmonic?

Let us be a bit more general and consider the situation in which the $\mathfrak{sp}(1)$ connection A appearing in Eq. (2.210) is *not* the pull-back of the $\mathfrak{sp}(1)$ connection, denoted B, defined on the hypervariety. By then differentiating Eq. (2.207), using Eqs. (2.210) and the formulas in App. (D.2), we obtain

$$\begin{aligned}\mathfrak{D}_m\partial_n q^X \;&=\; -2\varepsilon^{str}\left[\mathsf{A}^s{}_n \;-\; \partial_n q^Z\; \mathsf{B}^s{}_Z\right]\; \Phi^t{}_m{}^p\partial_p q^Y\; J^r{}_Y{}^X \\ &\quad +\Phi^r{}_n{}^p\; \mathfrak{D}_m\partial_p q^y\; \vec{J}^r{}_Y{}^X\,.\end{aligned} \tag{2.76}$$

Contracting the free indices, we find that

$$\mathfrak{D}_m\partial^m q^X = 2\varepsilon^{str}\left[\mathsf{A}^s{}_m - \partial_m q^Z\; \mathsf{B}^s{}_Z\right]\Phi^{t\;nm}\partial_n q^Y J^r{}_Y{}^X\,. \tag{2.77}$$

In our case, we have $\mathsf{A} = dq\cdot\mathsf{B}$ whence the fact that q is a quaternionic map, by itself, implies that it is harmonic.

Therefore, supersymmetric configurations of the hyperscalars consist of quaternionic maps q such that the $\mathfrak{su}(2)_-$ connection of the 4-dimensional space manifold is canceled by the pullback of the one on the hypervariety.

In the next section we are going to check whether the conditions that we have derived on the fields are sufficient to have unbroken supersymmetry, *i.e.* identically solve the KSEs.

Solving the Killing spinor equations

We begin with Eq. (2.46), from the gaugino supersymmetry transformation. After use of the expression of the vectorial fields Eq. (2.201), it can be put in the form

$$\left(2\, \partial\!\!\!/ \phi^x - \tfrac{\sqrt{3}}{2}\, \Theta\!\!\!\!/^I\right) R^- \epsilon^i = 0\,, \tag{2.78}$$

where we have defined the projectors $R^\pm$

$$R^\pm \equiv \tfrac{1}{2}\left(1 \pm \gamma^0\right)\,. \tag{2.79}$$

Obviously, this equation can always be solved by imposing the projection

$$R^- \epsilon^i = 0\,, \tag{2.80}$$

which is equivalent to a chirality condition on the spinors over the spatial manifold due to the relation $\gamma^0 = \gamma^{1234}$. R^+ and R^- have rank 2 and therefore this projection breaks/preserves 1/2 of the original supersymmetries.

Now we analyze Eq. (2.47), from the hyperinos supersymmetry transformations. Using Eq. (2.207) we can rewrite it in the form

$$f_X{}^{jA}\, \partial\!\!\!/ q^X \left[3\delta_j{}^i + \tfrac{i}{4}\sum_r J\!\!\!/^{(r)} \sigma^{(r)}{}_j{}^i \gamma_0\right] \epsilon_i - \gamma_m \mathsf{J}^r{}_{mn} \partial_n q^Y J^r{}_Y{}^X f_X{}^{iA} R^- \epsilon_i = 0\,, \tag{2.81}$$

which can be solved by imposing the projection Eq. (2.80) and

$$\Pi^{r+}{}_j{}^i \epsilon^j = 0\,, \tag{2.82}$$

where the $\Pi^{r\pm}{}_j{}^i$s are the objects defined in Eq. (2.214). The $\Pi^{r+}{}_j{}^i$ satisfy the algebra

$$\Pi^{r\,+}\Pi^{s\,+} = \tfrac{1}{2}\Pi^{r\,+} + \tfrac{1}{2}\Pi^{s\,+} - \tfrac{1}{2}\varepsilon^{rst}\Pi^{t\,+} - \tfrac{1}{4}\,\delta^{rs} R^-\,, \tag{2.83}$$

and are idempotent (and, therefore, projectors) only in the subspace of spinors satisfying the projection Eq. (2.80).

Observe that, in principle, we need to impose the three projections $r = 1, 2, 3$ on the Killing spinors. The above algebra shows that only two of them are independent and it is easy to see that they preserve only 1/4 of the supersymmetries preserved by the projection Eq. (2.80), *i.e.* only 1/8 of the supersymmetries is generically preserved in presence of non-trivial hyperscalars.

We turn now to Eq. (2.45) from the gravitino supersymmetry transformation. We consider separately the timelike and spacelike components of this equation. By using the spin connection of the five-dimensional metric Eqs. (C.17) and the expression of the vector fields Eq. (2.201), the timelike component takes the form

$$\partial_0 \epsilon^i + \left[2\, \not{\partial} f^{1/2} - \tfrac{1}{4} f \left(1 - \tfrac{1}{3}\gamma^0\right) \not{G}^+ - \tfrac{1}{4} f\, \not{G}^-\right] R^- \epsilon^i = 0\,, \tag{2.84}$$

which is automatically solved by time-independent Killing spinors satisfying the projection Eq. (2.80).

The space-like components of Eq. (2.45) take, after use of Eq. (2.80), the form

$$\nabla_m \eta^i \;+\; \eta^j \mathsf{A}_{m\,j}{}^i \;=\; 0\,, \qquad \eta^i \equiv f^{-1/2} \epsilon^i\,. \tag{2.85}$$

To solve this equation, the quaternionic nature of the 4-dimensional spatial manifold comes to our rescue: in the special Vierbein basis and $SU(2)$ gauge in which Eq. (2.211) holds, the 2-forms $\Phi^r{}_{mn}$ are the constants $\mathsf{J}^r{}_{mn}$. Using this splitting, the above equation takes the form

$$\nabla^+_m \eta^i \;+\; i\, \mathsf{A}^r{}_m \left(\sigma^r{}_j{}^i + \tfrac{i}{4}\, \not{\mathsf{J}}^r \delta_j{}^i\right) \eta^j = 0\,, \qquad \nabla^+_m \eta^i \;=\; (\partial_m \;+\; \tfrac{1}{4}\, \not{\xi}^+{}_m)\eta^i\,. \tag{2.86}$$

Using the projections Eq. (2.82) for each non-vanishing component of the pull-back of the $\mathfrak{su}(2)$ connection $\mathsf{A}^r{}_X \partial_m q^X$ we are left with

$$\nabla^+_m \eta^i = 0\,, \tag{2.87}$$

which is solved by constant spinors that satisfy the projection Eq. (2.80), *i.e.* if they are chiral in the 4-dimensional spaces of constant time.

It should be clear from the discussion of the gravitino variations, that, for some configurations, not all of the projections Π need be imposed, *e.g.* when turning on only an $\mathfrak{u}(1)$ in $\mathfrak{su}(2)_-$. The analysis of Eq. (2.81), however, indicates that still all 3 of the projections ought to be implemented. This is true if we disregard the possibility of a special coordinate dependency of the quaternionic map. As an extreme example we have the case with frozen hyperscalars which effectively is like not having them at all. A less-trivial example to this effect is fostered by the trivial uplift of the *c-mapped cosmic string* analyzed in [4, Sec. (4.4)], in which case the map is holomorphic.[5]

[5] In fact, part of Chen and Li's article [73] consists of showing that there are quaternionic maps between hyper-Kähler manifolds that are *not* holomorphic w.r.t. some complex structure.

Supersymmetric solutions

In Section 2.3 we proved that timelike supersymmetric configurations solve all the equations of motions if they solve the Maxwell equations and Bianchi identities which we rewrite here in differential form language for convenience:

$$4^{\star}\mathcal{E}_I = -d^{\star}\left(a_{IJ}F^J\right) + \tfrac{1}{\sqrt{3}}C_{IJK}F^J \wedge F^K\,, \tag{2.88}$$

$$\mathcal{B}^I = dF^I\,. \tag{2.89}$$

We may evaluate these expressions for supersymmetric configurations using the formula (2.201). The result is

$$\mathcal{E}_I^0 = -\frac{\sqrt{3}}{2}f^2\left[\nabla^2_{(4)}\left(h_I/f\right) - \tfrac{1}{4}C_{IJK}\Theta^J\cdot\Theta^K\right]\,, \tag{2.90}$$

$$\mathcal{E}_I^m = -2\sqrt{3}f^{3/2}C_{IJK}h^J(\star_{(4)}d\Theta^K)^m\,, \tag{2.91}$$

$$\left(\star\mathcal{B}^I\right)^{0m} = -\sqrt{3}f^{3/2}(\star_{(4)}d\Theta^I)^m\,. \tag{2.92}$$

where, as usual, all the objects in the r.h.s. of the equations are written in terms of the 4-dimensional spatial metric h. The components $\left(\star_{(4)}\mathcal{B}^I\right)^{mn}$ vanish identically, and it is immediate to see that the KSI Eq. (2.37) is satisfied.

Then, the supersymmetric solutions have to satisfy only these two equations:

$$\nabla^2_{(4)}\left(h_I/f\right) - \tfrac{1}{4}C_{IJK}\Theta^J\cdot\Theta^K = 0\,, \tag{2.93}$$

$$d\Theta^I = 0\,, \tag{2.94}$$

which are identical to those found in Ref. [72] in absence of hypermultiplets, the difference being the quaternionic nature of the 4-dimensional space.

2.4.3 Some explicit examples

In the recent paper Ref. [41] Jong, Kaya and Sezgin gave an explicit example with non-trivial and not-obviously-holomorphic hyperscalars taking values in the symmetric space $H_4 = SO(4,1)/SO(4)$. In this section we are going to use the same set-up to find 5-dimensional supersymmetric solutions and discuss the possible gravitational effects.

The four coordinates of the target are denoted by $q^{\underline{X}}$, $\underline{X} = 1, \ldots, 4$, and take the metric on the hypervariety to be

$$g_{\underline{X}\,\underline{Y}} = \Lambda^2 \delta_{\underline{XY}}\,, \qquad \Lambda(q^2) = \frac{1}{1\,-\,q^2}\,, \qquad q^2 \equiv q^{\underline{X}} q^{\underline{X}} \leq 1\,. \tag{2.95}$$

As one might have suspected this metric is Einstein, and since the space is conformally flat, it is also trivially selfdual, meaning that we are really dealing with an authentic 4-dimensional quaternionic-Kähler manifold.

A Vierbein for this metric is

$$E^X \;=\; \Lambda \delta^X{}_{\underline{Y}}\, dq^{\underline{Y}}\,, \qquad E_X \;=\; \Lambda^{-1} \delta_X{}^{\underline{Y}} \frac{\partial}{\partial q^{\underline{Y}}}\,. \tag{2.96}$$

In both the coordinate and the Vierbein basis the three complex structures are given by the 't Hooft symbols $\rho^r{}_{XY} (= \mathsf{J}^r_{XY})$, which are real, constant and antisymmetric matrices in the X, Y indices. Moreover they are anti-selfdual[6] and satisfy

$$\rho^r{}_{XY}\, \rho^s{}_{YZ} \;=\; -\delta^{rs}\, \delta_{XZ} \;+\; \epsilon^{rst}\, \rho^t{}_{XZ}\,, \tag{2.97}$$

$$\rho^r{}_{XY}\, \rho^r{}_{WZ} \;=\; \delta_{XW}\, \delta_{YZ} - \delta_{XZ}\, \delta_{YW} \;-\; \epsilon_{XYWZ}\,. \tag{2.98}$$

The anti-selfdual part of the spin connection is

$$\omega^{-XY} = 2\, \left(q^{[X} E^{Y]} - \tfrac{1}{2} \epsilon^{XYWZ} q^W E^Z \right)\,, \tag{2.99}$$

where $q^X \equiv \delta^X{}_{\underline{Y}}\, q^{\underline{Y}}$.

In order to construct the hyperscalars, we assume that also the base manifold is conformally flat, *i.e.*

$$h_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}} = \Omega^2 dx^{\underline{m}} dx^{\underline{m}}\,, \qquad \Omega = \Omega(x^2)\,, \qquad x^2 \equiv x^{\underline{m}} x^{\underline{m}}\,, \tag{2.100}$$

and thence take the Vierbein on the base manifold to be

$$V^m = \Omega \delta^m{}_{\underline{m}} dx^{\underline{m}}\,, \qquad V_m = \Omega^{-1} \delta_m{}^{\underline{m}} \partial_{\underline{m}}\,. \tag{2.101}$$

[6] They can be seen as the three anti-selfdual combinations of generators of $\mathfrak{so}(4)$, *i.e.* the generators of the $\mathfrak{su}(2)_-$ subalgebra.

In this basis we can identify the complex structures of the base manifold with those of the hypervariety

$$\mathsf{J}^r{}_m{}^n \;=\; \delta_m{}^X J^r{}_X{}^Y \delta^n_Y \;=\; \rho^r{}_{mn}\,. \tag{2.102}$$

The anti-selfdual part of the spin connection on the base manifold is

$$\xi^{-mn} = 2\frac{\Omega'}{\Omega^2}\left(x^{[m}V^{n]} - \tfrac{1}{2}\epsilon^{mnpq}x^pV^q\right) \tag{2.103}$$

where $x^m = \delta^m{}_{\underline{m}}\, x^{\underline{n}}$.

Now we analyze the conditions for supersymmetry on the hyperscalars $q^{\underline{X}}$. The first condition is that they must constitute a quaternionic map, *i.e.* Eq. (2.207), w.r.t. the chosen quaternionic structures. In our setting this equations takes the form

$$\partial_{\underline{m}}q^{\underline{X}} = \left(\delta_{\underline{mY}}\delta_{\underline{nX}} - \delta_{\underline{mX}}\delta_{\underline{nY}} - \epsilon_{\underline{mnYX}}\right)\partial_{\underline{n}}q^{\underline{Y}} \tag{2.104}$$

whose symmetric and antisymmetric parts give

$$\partial_{\underline{m}}q^{\underline{m}} \;=\; 0\,, \tag{2.105}$$

$$\partial_{[\underline{m}}q_{\underline{n}]} \;=\; -\tfrac{1}{2}\epsilon_{\underline{mnpq}}\partial_{\underline{p}}q_{\underline{q}}\,, \tag{2.106}$$

where $q_{\underline{m}} = q^{\underline{m}}$.

A solution to these equations is

$$q^{\underline{m}} \;=\; x^{\underline{m}}\,x^{-4}\,, \tag{2.107}$$

where we have chosen a possible multiplicative constant to be unity.

The second condition on the hyperscalars states that the anti-selfdual part of the spin connection of the base manifold must be related to the $\mathfrak{su}(2)$ connection induced from the target,

$$\xi^{-}_{mn}{}^{p} \;=\; -\vec{\mathsf{A}}_m\cdot\vec{\mathsf{J}}_n{}^p\,, \tag{2.108}$$

$$\vec{\mathsf{A}}_m \;\equiv\; \partial_m q^{\underline{X}}\,\vec{\omega}_{\underline{X}}\,, \tag{2.109}$$

where $\vec{\omega}_{\underline{X}}$ is the $\mathfrak{su}(2)$ connection of the target. We observe that the reasoning leading to the relation (2.108) can be applied on the target manifold as well,[7], where the

[7] Indeed it can be applied in any four-dimensional Riemannian manifold.

involved connections are ω_{XY} and $\vec{\omega}_{\underline{X}}$ and therefore we may establish the following relation on the target

$$\omega^{-}_{XY}{}^{Z} \;=\; -\vec{\omega}_{X} \;\cdot\; \vec{J}_{Y}{}^{Z}\,. \tag{2.110}$$

By contrasting Eqs. (2.108)-(2.110) we conclude that in our settings the anti-selfdual part of the spin connection of the base manifold is induced from the one of the hypervariety,

$$\xi^{-}_{m}{}^{np} \;=\; \partial_m q^{\underline{X}} \omega^{-}_{\underline{X}}{}^{YZ} \delta_{YZ}{}^{np}\,. \tag{2.111}$$

This condition is satisfied if

$$\frac{\Omega'}{\Omega} = \frac{1}{x^2\,(x^6-1)}\,. \tag{2.112}$$

The solution to this equation is

$$\Omega = \left(1 \;-\; x^{-6}\right)^{1/3}\,, \tag{2.113}$$

where, as above, we chose a certain multiplicative integration constant. We would like to point out that in this case the whole spin connection on the base manifold, rather than only its anti-selfdual part, is induced by the connection on the hypervariety.

A small investigation of the curvature invariants for the metric on the base space, shows that the point $x^2 = 1$ corresponds to a naked curvature singularity.

We have, thus, found the following 1/8 BPS, static, asymptotically flat, spherically symmetric, solution with only unfrozen hyperscalars in the $SO(1,4)/SO(4)$ coset:

$$\begin{aligned} ds^2 &= dt^2 - \left(1 - \frac{1}{x^6}\right)^{2/3} dx^{\underline{m}} dx^{\underline{m}}\,, \\ q^{\underline{m}} &= \frac{x^{\underline{m}}}{x^4}\,, \end{aligned} \tag{2.114}$$

which, as was said above, presents a naked singularity at $x^2 = 1$. Since there are no conserved charges in this system, the *no hair* conjecture suggests that black-hole type (i.e. spherically symmetric) solutions of this and similar systems will always be singular, but a more detailed study is needed to reach a final conclusion since they may be excluded by a mechanism like the one discussed in Ref. [74–76]. Furthermore, a higher-dimensional stringy interpretation of this, and similar solutions, is also needed as to interpret this singularity correctly.

As a further example let us now consider how solutions of minimal $N = 1, d = 5$ SUGRA[8] are deformed by the coupling to these hyperscalars. For the sake of simplicity we consider the simplest static ($\Theta = \omega = 0$) solution which is determined, according to Eq. (2.232), by a single function $f^{-1} = K$ which is harmonic w.r.t. the metric on the base manifold. The supersymmetric solution can be written as

$$\begin{aligned}
ds^2 &= K^{-2}\, dt^2 \;-\; K \left(1 + \frac{\lambda}{x^6}\right)^{2/3} dx^{\underline{m}} dx^{\underline{m}}\,, \\
A &= -\sqrt{3}\, K^{-1}\, dt\,, \\
q^{\underline{m}} &= \frac{x^{\underline{m}}}{x^4}\,.
\end{aligned} \tag{2.115}$$

If the harmonic function is chosen as to have an asymptotically flat, spherically symmetric solution with positive mass, the harmonic function is, with frozen hyperscalars,

$$K = 1 + \frac{|Q|}{x^2}\,, \tag{2.116}$$

and the solution is the 5-dimensional Reissner-Nordström black hole [77] which has an event horizon at $x = 0$ that covers all singularities.

When the hyperscalars are unfrozen and we have the above base manifold, K, determined again by imposing asymptotic flatness and spherical symmetry, is given by

$$K \;=\; 1 \;+\; Q\, \frac{{}_2\mathrm{F}_1\left(\frac{1}{3}, \frac{2}{3}\;;\; \frac{4}{3}\;;\; x^{-6}\right)}{x^2}\,, \tag{2.117}$$

where ${}_2F_1$ is a Gauß hypergeometric function. It is easy to see that $\lim_{x^2 \to \infty} K = 1$ and that ${}_2\mathrm{F}_1\left(\frac{1}{3}, \frac{2}{3}\;;\; \frac{4}{3}\;;\; x^{-6}\right)/x^2$ is a real, strictly positive and monotonically decreasing function on the interval $x^2 \in (1, \infty)$. The real question then is: what happens at $x^2 = 1$? Eq. [78, 15.1.20] gives a straightforward answer

$${}_2F_1\left(\, \tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; 1\,\right) \;=\; \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \;\sim\; 1.76664\,, \tag{2.118}$$

which implies that there is a naked singularity at $x^2 = 1$.

[8]In our notation this means that $n_v = 0$, $C_{111} = 1$ and $h^1 = 1$.

Solutions with an additional isometry

To make contact with the families of solutions with one additional isometry found in Refs. [35, 63] we make the following *Ansatz* for the 4-dimensional spacelike metric

$$h_{\underline{mn}}dx^m dx^n = H^{-1}(dz+\chi)^2 + H\gamma_{\underline{rs}}dx^r dx^s\,, \quad r,s=1,2,3\,, \tag{2.119}$$

where the function H, the 3-dimensional metric $\gamma_{\underline{rs}}$, and the 1-form $\chi = \chi_{\underline{r}}dx^r$ are all independent of the coordinate z. This *Ansatz* covers all 4-dimensional metrics with one isometry. We also require all fields in the solution to be independent of z.

As we have seen, supersymmetry requires the anti-selfdual part of the spin connection of this metric to be identical to the pullback of the $\mathfrak{su}(2)$ connection of the hypervariety. With the orientation $\varepsilon_{z123}=+1$ and the Vierbein basis

$$V^z = H^{-1/2}(dz+\chi)\,, \qquad V^r = H^{1/2}v^r\,, \tag{2.120}$$

where the v^r is the Dreibein for the 3-dimensional metric $\gamma_{\underline{rs}}$, the anti-selfdual part of the spin connection 1-form is given by

$$\begin{aligned} \xi^{-\,zr} &= \tfrac{1}{2}H^{-3/2}[\partial_r H - (\hat{\star}\hat{d}\chi)_r]V^z \\ &\quad +\tfrac{1}{4}\varepsilon_{rst}H^{-3/2}\{[\partial_t H - (\hat{\star}\hat{d}\chi)_t]\delta_{su} - 2H\varpi_{ust}\}V^u\,, \end{aligned} \tag{2.121}$$

where hatted objects refer to the 3-dimensional metric.

Observe that the z-independence of all fields means that the pullback of the $\mathfrak{su}(2)$ connection has no z component. Then, the supersymmetry condition Eq. (2.211) leads to

$$\hat{d}H = \hat{\star}\hat{d}\chi\,, \quad \Rightarrow \quad \hat{\nabla}^2 H = 0\,, \tag{2.122}$$

which is a condition on the 4-dimensional metric, and

$$\xi_{\underline{r}}^{-\,zs} \;=\; -\tfrac{1}{2}\varepsilon^{stu}\,\varpi_{\underline{r}}{}^{tu} \;=\; -2\mathsf{A}^s{}_X\,\partial_{\underline{r}}q^X\,, \tag{2.123}$$

which is a condition on the hyperscalars and the 3-dimensional metric.

Observe that the above 4-dimensional metric is a generalization of the Gibbons-Hawking instanton metric [64]. The non-trivial 3-dimensional metric destroys the selfduality of the connection. However, the generalized metric admits a quaternionic structure which is the straightforward generalization of that of the Gibbons-Hawking metric [79] and is, therefore, associated to the three hyper-Kähler 2-forms

$$J^r \equiv V^z \wedge V^r - \tfrac{1}{2}\varepsilon^{rst} V^s \wedge V^t \,. \tag{2.124}$$

It is trivial to check that they satisfy the quaternionic algebra since the tangent space components of these 2-forms are identical to those of the Gibbons-Hawking metric and are proportional to the anti-selfdual generators of $SO(4)$. Unlike the Gibbons-Hawking case, however, the hyper-Kähler 2-forms are not closed. Instead, a simple calculation shows that they satisfy

$$dJ^r - \varpi^{rs} \wedge J^s = 0 \,, \tag{2.125}$$

which, on account of Eq. (2.123), can be written in the form

$$dJ^r + 2\varepsilon^{rst}\mathsf{A}^s \wedge J^s = 0 \,. \tag{2.126}$$

Thus, the 4-dimensional metric Eq. (2.119) and hyperscalars subject to Eqs. (2.122) and (2.123) (plus Eq. (2.207)) are the most general ones associated to supersymmetric solutions with one isometry. Using them it can be shown that the general solutions found in Ref. [63] are formally identical, the only difference being that the $2\bar{n}+2$ harmonic functions K^I, L_I, M, H on which these solutions depend, are harmonic functions w.r.t. the 3-dimensional metric $\gamma_{\underline{rs}}$.

To be explicit, in terms of these harmonic functions, the scalars, the closed selfdual 2-forms Θ^I, and the 1-form ω take the form

$$\begin{aligned}
h_I/f &= C_{IJK}K^J K^K/H + L_I \,, \\
\Theta^I &= [(dz+\chi)\wedge d(K^I/H) + H\hat{\star} d(K^I/H)] \,, \\
\omega &\equiv \omega_5(d\psi+\chi) + \hat{\omega} \,, \\
\omega_5 &= M + \tfrac{3}{2}H^{-1}L_I K^I + H^{-2}C_{IJK}K^I K^J K^K \,, \\
\star_{(3)} d\hat{\omega} &= HdM - MdH + \tfrac{3}{2}(K^I dL_I - L_I dK^I) \,.
\end{aligned} \tag{2.127}$$

The function f has to be determined case by case using the constraint $C_{IJK}h^I h^J h^K = 1$, but an explicit expression for symmetric spaces is given in Ref. [63]. In the $n = 0$ case, *i.e.* only one function $K^0 \equiv K$ and one function $L_0 \equiv L$, it is given by

$$f^{-1} = K^2/H + L \,. \tag{2.128}$$

The metric of these solutions can be cast in the form

$$
\begin{aligned}
ds^2 &= -k^2[dz+B]^2 \\
&\quad +k^{-1}\left[\left(\tfrac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)(dt+\hat{\omega})^2-\left(\tfrac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)^{-1}\gamma_{\underline{rs}}dx^r dx^s\right], \\
k^2 &= f^{-1}H^{-1}-f^2\omega_5^2\,, \\
B &= \chi+f^2\omega_5 k^{-2}(dt+\hat{\omega})\,.
\end{aligned}
\tag{2.129}
$$

In this form, comparing with the results of Refs. [3,4] it is easy to see the form of the $N=2, d=4$ supersymmetric solution that will appear after dimensional reduction. The metric

$$
ds^2=\left(\frac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)(dt+\hat{\omega})^2-\left(\frac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)^{-1}\gamma_{\underline{rs}}dx^r dx^s\,,
\tag{2.130}
$$

is that of a solution in the timelike class, to which all $N=2, d=4$ supersymmetric black holes belong, and there is an additional scalar (k) and an additional vector field (B). If the 5-dimensional solution is static $\omega_5=0$ and the vector field $B=\chi$ is magnetic and corresponds to a KK monopole or a generalization thereof. This fact has been used in Refs. [42–48] to relate 4- and 5-dimensional black hole solutions.

2.4.4 The null case

Denote the null Killing vector by l^μ. Following the same considerations as in Refs. [35, 80], we find that we can choose null coordinates u and v such that

$$
l_\mu dx^\mu = f du\,,\qquad l^\mu\partial_\mu=\partial_{\underline{v}}\,,
\tag{2.131}
$$

where f may depend on u but not on v, and the metric can be put in the form

$$
ds^2=2fdu(dv+Hdu+\omega)-f^{-2}\gamma_{\underline{rs}}dx^r dx^s\,,
\tag{2.132}
$$

where $r,s,t=1,2,3$ and the 3-dimensional spatial metric $\gamma_{\underline{rs}}$ may also depend on u but not on v. Eqs. (2.190) and (2.197) state that the scalars are v-independent.

The above metric is completely equivalent to the one used in Refs. [35, 80], but we find this form more convenient; a Vielbein, and the corresponding spin connection and curvature for it are given in Appendix C.4.

In the null case the Fierz identities (B.99,B.100) and (B.101) imply that the 2-forms bilinears Φ^r are of the form

$$\Phi^r = du \wedge v^r \,, \tag{2.133}$$

where the 1-forms v^r are an orthogonal basis for the 3-dimensional spatial metric $\gamma_{\underline{rs}}$. Eq. (2.189) then implies the equation

$$du \wedge Dv^r = 0 \,, \tag{2.134}$$

i.e. the spatial components of the pullback of the $\mathfrak{su}(2)$ connection are related to the spin connection coefficients of the basis v^r (computed for constant u) by

$$\varpi_{\underline{r}}{}^{st} \;=\; 2\varepsilon^{stp}\, \mathsf{A}^p{}_X\, \partial_{\underline{r}} q^X \,. \tag{2.135}$$

This equation is identical to the one found in Ref. [4] in the context of ungauged $N = 2, d = 4$ supergravity coupled to hypermultiplets. Actually, substituting the 2-forms we found into Eq. (2.195) we arrive at

$$\partial_r q^X f_X{}^{iA} \sigma^r{}_i{}^j = 0 \,, \tag{2.136}$$

which is identical to the equation that the hyperscalars have to satisfy in a supersymmetric configuration of ungauged $N = 2, d = 4$ supergravity [4]. Observe that the last two equations together with Eq. (D.30) (for $\nu = -1$) imply that the Ricci scalar of the 3-dimensional metric γ satisfies

$$R_{rs}(\gamma) = -\tfrac{1}{2} g_{XY} \partial_r q^X \partial_s q^Y \,. \tag{2.137}$$

Let us now determine the vector field strengths: Eqs. (2.48,2.191) and (2.193) lead to

$$l^\mu F^I_{\mu\nu} = 0 \,, \tag{2.138}$$

and, using the basis given in Appendix C.4, we can write

$$F^I = F^I{}_{+r} e^+ \wedge e^r + \tfrac{1}{2} F^I{}_{rs} e^r \wedge e^s = F^I{}_{+r} du \wedge v^r + \tfrac{1}{2} f^{-2} F^I{}_{rs} v^r \wedge v^s \,. \tag{2.139}$$

From Eq. (2.186) we get[9]

[9] Unless stated otherwise (as is the case of $F^I{}_{rs}$) all quantities with flat spatial indices refer to the 3-dimensional metric and Dreibein basis.

$$h_I F^I{}_{rs} = -\sqrt{3}\varepsilon_{rst}\partial_t f\,, \qquad \partial_t \equiv v_t{}^{\underline{s}}\partial_{\underline{s}}\,. \tag{2.140}$$

The same result can be obtained from $D\star\Phi^r$. From Eq. (2.194) we get

$$h_I^x F^I{}_{rs} \;=\; -\varepsilon_{rst} f\;\partial_t\phi^x\,, \tag{2.141}$$

which, together with the previous equation and the definition of h_I^x give

$$f^{-2}\;F^I{}_{rs} \;=\; \sqrt{3}[\hat{\star}\;\hat{d}(h^I/f)\;]_{rs}\,. \tag{2.142}$$

From the $++r$ components of Eq. (2.187) we get

$$h_I F^I{}_{+r} = -\tfrac{1}{\sqrt{3}} f^2(\hat{\star}F)_r\,, \tag{2.143}$$

where

$$F = \hat{d}\omega\,. \tag{2.144}$$

The components $h_I^x F^I{}_{+r}$ are not determined by supersymmetry and we parametrize them by 1-forms ψ^I satisfying $h_I\psi^I = 0$. In conclusion, the vector field strengths are given by

$$F^I = [\tfrac{1}{\sqrt{3}} f^2 h^I \hat{\star}F - \psi^I]\wedge du + \sqrt{3}\hat{\star}\hat{d}(h^I/f)\,. \tag{2.145}$$

Solving the Killing spinor equations

Let us continue our analysis by plugging our configuration into Eq. (2.46): using the Vielbein, Eq. (2.246) and some Clifford algebra manipulations, we see that

$$0 \;=\; f^{-1}\left[\partial_u\phi^x \;+\; h_I^x\psi_r^I\;\gamma^r \;+\; \tfrac{f^2}{2}\partial_t\phi^x\;\varepsilon_{trs}\gamma^{rs}\gamma^-\right]\;\gamma^+\epsilon^i\;, \tag{2.146}$$

so, if we want the scalars ϕ and the ψ^I to be non-trivial, we are forced to impose $\gamma^+\epsilon^i \;=\; 0$.

As is usual in wave-like supersymmetric solutions, the $-$ component of the susy variation (2.45) is identically satisfied by an v-independent spinor, and the remainder of the components simplify greatly due to the lightlike constraint: The ones in the r-directions reduce, after using Eqs. (2.245,2.248), to

$$
\begin{aligned}
0 &= f\,\mathsf{D}_r\epsilon = f\left[\partial_r - \tfrac{1}{4}\varpi_{rst}\gamma^{st} + i\vec{\mathsf{A}}\cdot\vec{\sigma}^T\right]\epsilon \\
&= f\left[\partial_r + \mathsf{A}^p_r\gamma^p\left(1 - i\gamma^p(\sigma^{(p)})^T\right)\right]\epsilon\,,
\end{aligned}
\tag{2.147}
$$

where in the last step we made use of Eq. (2.240). If we then introduce the projection operators (no sum over p!)

$$
\Pi_p = \tfrac{1}{2}\left(1 - i\gamma^p(\sigma^{(p)})^T\right) \quad ; \quad \Pi_p^2 = \Pi_p \quad ; \quad [\,\Pi_p\,,\,\Pi_q\,] = 0\,, \tag{2.148}
$$

the above equation is solved by imposing the condition $\Pi_p\epsilon = 0$, for every p for which A^p does not vanish, leading to a Killing spinor that can only depend on u.

The penultimate equation that needs to be checked is the gravitino variation in the u-direction.

$$
0 = \partial_u\epsilon + \tfrac{1}{4}v_r{}^{\underline{t}}\partial_u v_{s\underline{t}}\,\gamma^{rs}\epsilon + i\vec{\mathsf{A}}_u\cdot\vec{\sigma}^T\epsilon = \partial_u\epsilon - \left[\mathsf{A}^p{}_u + \tfrac{1}{4}\varepsilon_{prs}v_r{}^{\underline{t}}\partial_u v_{s\underline{t}}\right]\gamma^p\epsilon\,. \tag{2.149}
$$

Generically the factor $v_r{}^{\underline{t}}\partial_u v_{s\underline{t}}$ is spacetime dependent, which, in order to avoid an inconsistency with the x-independency of the Killing spinor, means that we must have

$$
\mathsf{A}^p{}_u = -\tfrac{1}{4}\varepsilon_{prs}\,v_r{}^{\underline{t}}\,\partial_u v_{s\underline{t}}\,. \tag{2.150}
$$

A consequence of this analysis is that the Killing spinor is constant.

Eq. (2.47) is the only one left to be analyzed. In fact it is straightforward to see that, given the constraints obtained thus far, Eq. (2.47) is tantamount to (2.136) contracted with ϵ_j. In order to get this far, however, one has to make use of all the constraints, meaning that if we do not want even more constraints, Eq. (2.136) must hold.

Equations of motion

In the null case, the KSIs contain far less restrictive information than in the timelike case, and as one can see from Eqs. (2.40)-(2.44), there are more equations of motion to be checked.

In order to get on with the show, let us analyze the gauge sector: the non-vanishing components of the Bianchi identities are immediately found to be

$$
\begin{aligned}
{}^{\star}\mathcal{B}^{I\,+-} &= \sqrt{3}f^{3}\hat{\nabla}^{2}(h^{I}/f)\,, &(2.151)\\
f^{-1\star}\mathcal{B}^{I\,-r} &= [\hat{\star}\hat{d}(\tfrac{1}{\sqrt{3}}f^{2}h^{I}\hat{\star}F-\psi^{I})]_{r}+\sqrt{3}\left[\hat{\star}\partial_{\underline{u}}\hat{\star}\hat{d}(h^{I}/f)\right]_{r}\,, &(2.152)
\end{aligned}
$$

and the Maxwell equations take the form

$$
4^{\star}\mathcal{E}_{I}=-\sqrt{3}du\wedge\left\{f\hat{d}h_{I}\wedge F+\tfrac{1}{\sqrt{3}}\left[\hat{d}(\hat{\star}\psi_{I}/f)-2C_{IJK}\psi^{J}\wedge\hat{\star}\hat{d}(h^{K}/f)\right]\right\}\,, \quad (2.153)
$$

and satisfy the KSIs Eqs. (2.41) and (2.42).

Eq. (2.151) is solved by $\bar{n}\equiv n_{v}+1$ harmonic[10] functions K^{I}:

$$
h^{I}/f = K^{I}\quad,\quad \hat{\nabla}^{2}\,K^{I} = 0\,, \quad (2.154)
$$

$K^{I}\neq 0$, which, as in the timelike case, determines f to be

$$
f^{-3} = K_{I}K^{I}\,,\quad K_{I}\equiv C_{IJK}K^{J}\,K^{K}\,. \quad (2.155)
$$

Since the K^{I} are harmonic, we may introduce $\bar{n}$ local, 3-dimensional 1-forms $\alpha^{I}=\alpha^{I}_{\underline{r}}(u,\vec{x})dx^{r}$ which satisfy

$$
\hat{d}\alpha^{I}=\hat{\star}\hat{d}K^{I}\,, \quad (2.156)
$$

such that each α^{I} is determined, up to a 3-dimensional gradient, in terms of K^{I} and γ. This gauge freedom will be relevant soon.

Eqs. (2.152) become

$$
\hat{d}\psi^{I}=\tfrac{1}{\sqrt{3}}\hat{d}\left(f^{2}h^{I}\hat{\star}F\right)+\sqrt{3}\hat{d}\dot{\alpha}^{I}\,, \quad (2.157)
$$

where $\dot{\alpha}\equiv\dot{\alpha}^{I}_{\underline{r}}\,dx^{r}$. The general, local solution to this equation is

$$
\psi^{I}=\tfrac{1}{\sqrt{3}}f^{2}h^{I}\hat{\star}F+\hat{d}M^{I}+\sqrt{3}\dot{\alpha}^{I}\,, \quad (2.158)
$$

where the M^{I}s are some functions. The constraint $h\cdot\psi=0$ implies

$$
\tfrac{1}{\sqrt{3}}f^{2}\hat{\star}F+h_{I}\hat{d}M^{I}+\sqrt{3}h_{I}\dot{\alpha}^{I} = 0\,. \quad (2.159)
$$

[10]In this section, harmonic means harmonic on the 3-dimensional Euclidean space with metric γ.

Due to the relation $F = \hat{d}\omega$, the above is the equation that, if we manage to fix the Ms, will determine ω.

Plugging Eq. (2.158) into the Maxwell equations we see that

$$\hat{\nabla}^2 L_I \;+\; \sqrt{3}C_{IJK}\left[\hat{\nabla}_r\left(K^J\dot{\alpha}^K\right)_r + \partial_r K^J\left(\dot{\alpha}^K\right)_r\right] \;=\; 0\,, \tag{2.160}$$

where we have defined the combinations

$$L_I \;\equiv\; C_{IJK}\; K^J\; M^K\,. \tag{2.161}$$

At this point we take advantage of the gauge freedom of (2.156) in order to simplify the Maxwell equations: fix the gauge by imposing

$$C_{IJK}\left[\hat{\nabla}_r\left(K^J\dot{\alpha}^K\right)_r + \partial_r K^J\left(\dot{\alpha}^K\right)_r\right] \;=\; 0\,, \tag{2.162}$$

thus determining α^I completely in terms of the K^I and γ. In this gauge the functions L_I are harmonic,

$$\hat{\nabla}^2\; L_I \;=\; 0\,, \tag{2.163}$$

and we determine the functions M^I in terms of the harmonic functions K^I and L_I by Eq. (2.161).

Another advantage of the above gauge is that the equation for ω, Eq. (2.159), takes on the rather nice form:

$$\hat{\star}\hat{d}\omega \;=\; \sqrt{3}\left(L_I dK^I \;-\; K^I dL_I\right) \;-\; 3K_I\dot{\alpha}^I\,. \tag{2.164}$$

In the analysis of the Einstein equations it is useful to perform the following change of variables

$$H \;=\; -\tfrac{1}{2}L_I M^I \;+\; N\,. \tag{2.165}$$

With this redefinition $\mathcal{E}_{++}$ becomes

$$\begin{aligned}\mathcal{E}_{++} \;=\;& -f\nabla^2 N + f\Big[\nabla_r(\dot{\omega})_r + 3(\dot{\omega})_r\partial_r\log f + \tfrac{1}{2}f^{-3}(\ddot{\gamma})_{rr} + \tfrac{1}{4}f^{-3}(\dot{\gamma})^2 - \tfrac{3}{2}f^{-4}\dot{f}(\dot{\gamma})_{rr}\\ &-3C_{IJK}K^I\left(\dot{K}^J\dot{K}^K + (\dot{\alpha}^J)_r(\dot{\alpha}^K)_r + \tfrac{2}{\sqrt{3}}(\dot{\alpha}^J)_r\partial_r M^K\right) + 12f^3\left(K_I\dot{K}^I\right)^2\\ &+\tfrac{1}{2}f^{-3}g_{XY}\dot{q}^X\dot{q}^Y\Big]\,.\end{aligned} \tag{2.166}$$

In general there is a gauge freedom in setting the one-form ω given in (2.164), corresponding to shifts in the coordinate v. If we choose to fix this gauge freedom by demanding

$$
\begin{aligned}
\nabla_r(\dot{\omega})_r + 3(\dot{\omega})_r\partial_r \log f &= -\tfrac{1}{2}f^{-3}(\ddot{\gamma})_{rr} - \tfrac{1}{4}f^{-3}(\dot{\gamma})^2 + \tfrac{3}{2}f^{-4}\dot{f}(\dot{\gamma})_{rr} - \tfrac{1}{2}f^{-3}g_{XY}\dot{q}^X\dot{q}^Y \\
&\quad +3C_{IJK}K^I\left(\dot{K}^J\dot{K}^K + (\dot{\alpha}^J)_r(\dot{\alpha}^K)_r + \tfrac{2}{\sqrt{3}}(\dot{\alpha}^J)_r\partial_r M^K\right) \\
&\quad -12f^3\left(K_I\dot{K}^I\right)^2 , \qquad (2.167)
\end{aligned}
$$

then $\mathcal{E}_{++}$ vanishes identically if N is a real, harmonic function. $\mathcal{E}_{+r}$ becomes

$$
\mathcal{E}_{+r} = -\tfrac{1}{2}\nabla_s(\dot{\gamma})_{rs} + \tfrac{1}{2}\partial_r(\dot{\gamma})_{ss} + \tfrac{3}{2}f^3\dot{K}_I\partial_r K^I + \tfrac{1}{2}g_{XY}\dot{q}^X\partial_r q^Y , \qquad (2.168)
$$

whereas $\mathcal{E}_{rs}$ is identically satisfied by the configuration as we have it.

u-independent solutions

The equations that need to be solved, simplify greatly if we consider the case that the solutions do not depend on the coordinate u: in that case the gauge-fixings Eqs. (2.162,2.269) and the remaining equation of motion, Eq. (2.168), vanish identically, meaning that now the solutions are completely determined by the hyperscalars, the 3-dimensional metric and the $2\bar{n}+1$ real, harmonic functions L_I, K^I and N. Given these ingredients, in order to fully specify the solution we need calculate f, H, ω and ψ^I through the following, simplified equations.

$$
\begin{aligned}
f^{-3} &= K_I\,K^I , & L_I &= C_{IJK}K^J\,M^K , \\
H &= -\tfrac{1}{2}L_I\,M^I + N , & \hat{\star}\hat{d}\omega &= \sqrt{3}\left[L_I\hat{d}K^I - K^I\hat{d}L_I\right] , \\
h^I(\phi) &= f\,K^I , & \psi^I &= f^3K^I(L_J\hat{d}K^J - K^J\hat{d}L_J) + \hat{d}M^I .
\end{aligned}
\qquad (2.169)
$$

Solutions that belong to this family, but depending on a smaller number of harmonic functions have been given *e.g.* in Refs. [81–84].

Apart from being one of the nicest subclasses of solutions, the u-independent one becomes doubleplus interesting when we observe that if we reduce a solution in the null class over the spacelike direction $\sqrt{2}y = u - v$, which implies u-independence, we end up with a solution in the timelike class of $N = 2$ $d = 4$ SUGRA. In fact, comparing the constraints in this section with the ones in [4, Sec. (5)], one finds the same constraints on the hyperscalars and the 3-dimensional metric.

The metric Eq. (2.235) can be put in an y-adapted system, and one finds

$$
\begin{aligned}
ds^2 &= -k^2[dy + A]^2 + k^{-1}\left[\left(\tfrac{f^3}{1-H}\right)^{1/2}(dt + \tfrac{1}{\sqrt{2}}\omega)^2 - \left(\tfrac{f^3}{1-H}\right)^{-1/2}\gamma_{\underline{rs}}dx^r dx^s\right], \\
k^2 &= (1-H)f\,, \\
A &= -(1-H)^{-1}(Hdt + \tfrac{1}{\sqrt{2}}\omega)\,.
\end{aligned}
\tag{2.170}
$$

The 4-dimensional solutions can be easily read from these. Apart from the scalar k and the vector field A, which is purely electric if the 5-dimensional solution is static ($\omega = 0$), the metric takes the form

$$
ds^2 = \left(\frac{f^3}{1-H}\right)^{1/2}(dt + \tfrac{1}{\sqrt{2}}\omega)^2 - \left(\frac{f^3}{1-H}\right)^{-1/2}\gamma_{\underline{rs}}dx^r dx^s\,, \tag{2.171}
$$

and belongs to the $N = 2, d = 4$ timelike class to which all black-hole-type solutions belong in $d = 4$.

This 4-dimensional solution should be compared to the one in Eq. (2.129), which is the one one obtains when imposing an extra isometry on the four dimensional spacelike manifold in the timelike case. the main difference between them is the electric or magnetic nature of the KK vector field. In the simplest case this solutions would give a 4-dimensional electric KK black hole and the other one a 4-dimensional magnetic KK black hole, related by 4-dimensional electric-magnetic duality, as we discussed in the introduction. In the more general case, the relation between these solutions is more complicated and we hope to say more about it in the near future.

2.5 Gauged $N = 1$, $d = 5$ Supergravity

2.5.1 $N = 1, d = 5$ supergravity with gaugings

In this section we are going to briefly describe the action, equations of motion and supersymmetry transformation rules of gauged $N = 1, d = 5$ supergravities, which we take from Ref. [65], relying in the description of the ungauged theories given in

Ref. [85], whose conventions we follow. Appendix E contains a description of the gauging of the isometries of the scalar manifolds of the theory in which the definitions of the covariant derivatives $\mathfrak{D}$, gauge transformations and momentum map $\vec{P}_I$ can be found.

The bosonic action of $N=1, d=5$ gauged supergravity is given by

$$\begin{aligned} S \;=\; & \int d^5x\sqrt{g}\,\Big\{R+\tfrac{1}{2}g_{xy}\mathfrak{D}_\mu\phi^x\mathfrak{D}^\mu\phi^y+\tfrac{1}{2}g_{XY}\mathfrak{D}_\mu q^X\mathfrak{D}^\mu q^Y+\mathcal{V}(\phi,q)-\tfrac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu} \\ & +\tfrac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}\left(F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha-\tfrac{1}{2}gf_{LM}{}^IF^J{}_{\mu\nu}A^K{}_\rho A^L{}_\sigma A^M{}_\alpha\right. \\ & \left.+\tfrac{1}{10}g^2f_{LM}{}^If_{NP}{}^JA^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\alpha\right)\Big\}\,, \end{aligned} \tag{2.172}$$

where

$$\mathcal{V}(\phi,q)=g^2\left(4C_{IJK}h^I\vec{P}^J\cdot\vec{P}^K-\tfrac{3}{2}h^Ih^Jk_I{}^Xk_J{}^Yg_{XY}\right)\,, \tag{2.173}$$

is the potential for the scalars. In the limit of pure supergravity, $n_H=n_V=0$, $\mathcal{V}$ becomes a cosmological constant.

The equations of motion, for which we use the same notation as in Ref. [85], are

$$\begin{aligned} \mathcal{E}_{\mu\nu} \;=\; & G_{\mu\nu}-\tfrac{1}{2}a_{IJ}\left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho}-\tfrac{1}{4}g_{\mu\nu}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\right) \\ & +\tfrac{1}{2}g_{xy}\left(\mathfrak{D}_\mu\phi^x\mathfrak{D}_\nu\phi^y-\tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho\phi^x\mathfrak{D}^\rho\phi^y\right) \\ & +\tfrac{1}{2}g_{XY}\left(\mathfrak{D}_\mu q^X\mathfrak{D}_\nu q^Y-\tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho q^X\mathfrak{D}^\rho q^Y\right)-\tfrac{1}{2}g_{\mu\nu}\mathcal{V}\,, \end{aligned} \tag{2.174}$$

$$g^{xy}\mathcal{E}_y \;=\; \mathfrak{D}_\mu\mathfrak{D}^\mu\phi^x+\tfrac{1}{4}\partial^x a_{IJ}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}-\partial^x\mathcal{V} \tag{2.175}$$

$$g^{XY}\mathcal{E}_Y \;=\; \mathfrak{D}_\mu\mathfrak{D}^\mu q^X-\partial^X\mathcal{V}\,, \tag{2.176}$$

$$\mathcal{E}_I{}^\mu \;=\; \mathfrak{D}_\nu F_I{}^{\nu\mu}+\tfrac{1}{4\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}C_{IJK}F^J{}_{\nu\rho}F^K{}_{\sigma\alpha}+g\left(k_{I\,x}\mathfrak{D}^\mu\phi^x+k_{I\,X}\mathfrak{D}^\mu q^X\right)\,. \tag{2.177}$$

The supersymmetry transformation rules for the fermionic fields, evaluated on vanishing fermions, are

$$\delta_\epsilon \psi^i_\mu = \mathfrak{D}_\mu \epsilon^i - \tfrac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta} \left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta\right)\epsilon^i + \tfrac{1}{2\sqrt{3}} g h^I \gamma_\mu \epsilon^j P_{I\,j}{}^i\,, \tag{2.178}$$

$$\delta_\epsilon \lambda^{ix} = \tfrac{1}{2}\left(\not{\mathfrak{D}}\phi^x - \tfrac{1}{2} h^x_I \not{F}^I\right)\epsilon^i + g h^x_I \epsilon^j P^I{}_j{}^i\,, \tag{2.179}$$

$$\delta_\epsilon \zeta^A = \tfrac{1}{2} f_X{}^{iA}\left(\not{\mathfrak{D}} q^X + \sqrt{3} g h^I k_I{}^X\right)\epsilon_i\,. \tag{2.180}$$

The supersymmetry transformation rules of the bosonic fields are exactly the same as in the ungauged case [85]. This implies that the form of the Killing spinor identities (KSIs) relating the bosonic equations of motion that one can derive from them [6, 71] have the same form as in the ungauged case, given in [85], although the equations of motion are now those given above, which differ from those of the ungauged case by g-dependent terms.

2.5.2 Supersymmetric configurations and solutions

Following the standard procedure, we assume that the KSEs

$$\mathfrak{D}_\mu \epsilon^i - \tfrac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta\right)\epsilon^i + \tfrac{1}{2\sqrt{3}} g \gamma_\mu \epsilon^j h^I P_{I\,j}{}^i = 0\,, \tag{2.181}$$

$$\left(\not{\mathfrak{D}}\phi^x - \tfrac{1}{2} h^x_I \not{F}^I\right)\epsilon^i + 2 g \epsilon^j h^x_I P^I{}_j{}^i = 0\,, \tag{2.182}$$

$$f_X{}^{iA}\left(\not{\mathfrak{D}} q^X + \sqrt{3} g h^I k_I{}^X\right)\epsilon_i = 0\,, \tag{2.183}$$

admit at least one solution ϵ^i and we start deriving from them the equations satisfied by the tensor bilinears that can be constructed from the Killing spinor: the scalar f, the vector V (both $SU(2)$ singlets) and the three 2-forms Φ^r, which form an $SU(2)$-triplet.

The fact that the Killing spinor satisfies Eq. (2.181) leads to the following differential equations for the bilinears:

$$df = \tfrac{1}{\sqrt{3}} h_I i_V F^I\,, \tag{2.184}$$

$$\nabla_{(\mu} V_{\nu)} = 0\,, \tag{2.185}$$

$$
\begin{aligned}
dV &= -\tfrac{2}{\sqrt{3}} f h_I F^I - \tfrac{1}{\sqrt{3}} h_I{}^\star \left(F^I \wedge V\right) - \tfrac{2}{\sqrt{3}} g h^I \vec{P}_I \cdot \vec{\Phi}\,, && (2.186)\\
\mathfrak{D}_\alpha \vec{\Phi}_{\beta\gamma} &= -\tfrac{1}{\sqrt{3}} h_I F^{I\,\rho\sigma} \left(g_{\rho[\beta}{}^\star\vec{\Phi}_{\gamma]\alpha\sigma} - g_{\rho\alpha}{}^\star\vec{\Phi}_{\beta\gamma\sigma} - \tfrac{1}{2} g_{\alpha[\beta}{}^\star\vec{\Phi}_{\gamma]\rho\sigma}\right)\\
&\quad + \tfrac{1}{\sqrt{3}} g h^I \left(\vec{P}_I \times {}^\star\vec{\Phi}_{\alpha\beta\gamma} + 2 g_{\alpha[\beta} V_{\gamma]} \vec{P}_I\right)\,, && (2.187)
\end{aligned}
$$

where

$$
\mathfrak{D}_\alpha \vec{\Phi}_{\beta\gamma} = \nabla_\alpha \vec{\Phi}_{\beta\gamma} + 2\vec{B}_\alpha \times \vec{\Phi}_{\beta\gamma}\,. \tag{2.188}
$$

The differential equation for Φ^r (2.187) implies

$$
d\Phi^r + 2\varepsilon^{rst} B^s \wedge \Phi^t = \sqrt{3} g h^I \epsilon^{rst} P_I^s\, {}^\star\Phi^t\,. \tag{2.189}
$$

The fact that the Killing spinor satisfies Eqs. (2.182) and (2.183) leads to the following algebraic equations for the tensor bilinears:

$$
\begin{aligned}
V^\mu \mathfrak{D}_\mu \phi^x &= 0\,, && (2.190)\\
h_I^x F^I_{\alpha\beta} \vec{\Phi}^{\alpha\beta} &= 4 g f h_I^x \vec{P}^I\,, && (2.191)\\
V^\mu \mathfrak{D}_\mu q^X &= -\sqrt{3} g f h^I k_I{}^X\,, && (2.192)\\
f \mathfrak{D}_\mu \phi^x - h_I^x F^I{}_{\mu\nu} V^\nu &= 0\,, && (2.193)\\
\vec{\Phi}_{\mu\nu} \mathfrak{D}^\nu \phi^x + \tfrac{1}{4} \epsilon_{\mu\nu\alpha\beta\gamma} h_I^x F^{I\,\nu\alpha} \vec{\Phi}^{\beta\gamma} &= -2 g h_I^x \vec{P}^I V_\mu\,, && (2.194)\\
f \mathfrak{D}_\mu q^X + \Phi^r{}_\mu{}^\nu \mathfrak{D}_\nu q^Y J^r{}_Y{}^X &= -\sqrt{3} g h^I k_I{}^X V_\mu\,. && (2.195)
\end{aligned}
$$

We are now ready to extract consequences of these equations. To start with, Eq. (2.185) says that V is an isometry of the space-time metric. It is convenient to partially fix the G gauge using the condition

$$
i_V A^I + \sqrt{3} f h^I = 0\,, \tag{2.196}
$$

since then Eqs. (2.192) and (2.190) become just

$$
\mathcal{L}_V q^X = \mathcal{L}_V \phi^x = 0\,, \tag{2.197}
$$

after use of the explicit expression of the Killing vectors $k_I{}^x$ Eq. (E.6). Then, in this gauge, the scalars q^X, ϕ^x and f are independent of the coordinate adapted to the isometry (see Eq. (2.184).

We now consider separately the timelike ($f \neq 0$) and null ($f = 0$) cases.

The timelike case

The equations for the bilinears

By definition this is the case in which V^μ is timelike, $V^2 = f^2 > 0$. Introducing an adapted time coordinate t: $V = \partial_t$ the metric can be written in the same form as in the ungauged case:

$$ds^2 = f^2 \left(dt + \omega\right)^2 - f^{-1} h_{\underline{mn}} dx^m dx^n\,, \tag{2.198}$$

with ω and $h_{\underline{mn}}$ independent of time. As we mentioned in the previous section, in the (partially) fixed G-gauge ($A^I{}_t = -\sqrt{3} f h^I$) f, ϕ^x and q^X are also time-independent.

The spatial metric $h_{\underline{mn}}$ is endowed with an *almost* quaternionic structure, $\Phi^r{}_{\underline{m}}{}^{\underline{n}}$. This is an algebraic property that only depends on the Fierz identities.

The next step is to obtain the form of the supersymmetric vector field strength from Eqs. (2.184), (2.186), (2.191) and (2.193). In order to write the result it is convenient to split the gauge potential A^I into an electric part, which is determined by the partial gauge fixing $A^I{}_t = -\sqrt{3} f h^I$ and a magnetic part $\hat{A}^I$ with only spatial components

$$A^I = -\sqrt{3} h^I e^0 + \hat{A}^I\,, \tag{2.199}$$

$$A^I{}_{\underline{m}} = \hat{A}^I{}_{\underline{m}} - \sqrt{3} f h^I \omega_{\underline{m}}\,. \tag{2.200}$$

Observe that, unlike the spatial components $A^I{}_{\underline{m}}$, the components $\hat{A}^I{}_{\underline{m}}$ are invariant under local shifts of the time coordinate: $t \to t + \delta t(x)$, $\omega \to \omega - d\delta t(x)$ which do not change the form of the metric and, in particular, leave the 4-dimensional metric $h_{\underline{mn}}$ invariant. It is the correct 4-dimensional potential in the Kaluza-Klein sense.

In terms of the new variables $\hat{A}^I$ the field strengths are given by

$$F^I = -\sqrt{3}\, \hat{\mathfrak{D}}(h^I e^0) + \hat{F}^I\,, \tag{2.201}$$

where $\hat{\mathfrak{D}}$ is the 4-dimensional spatial covariant derivative[11] with respect to $\hat{A}^I$ and $\hat{F}^I$ is the non-Abelian field strength of $\hat{A}^I$ and it is related to ω and the scalars by

$$h_I\hat{F}^{I+} = \tfrac{2}{\sqrt{3}}(fd\omega)^+\,, \tag{2.202}$$

$$\hat{F}^{I-} = -2gf^{-1}C^{IJK}h_J\vec{P}_K\cdot\vec{\Phi}\,. \tag{2.203}$$

$\tilde{F}^{I+}$ is related to the 2-forms called Θ^I in the ungauged case [35, 80, 85] by

$$\Theta^I = -\tfrac{1}{\sqrt{3}}\hat{F}^{I+}\,. \tag{2.204}$$

It is also convenient to introduce the spatial $SU(2)$ connection $\hat{\vec{B}}$

$$\hat{\vec{B}} \equiv \vec{A} + \tfrac{1}{2}g\hat{A}^I\vec{P}_I\,, \tag{2.205}$$

$$\vec{B} = -\tfrac{\sqrt{3}}{2}h^I\vec{P}_Ie^0 + \hat{\vec{B}}\,, \tag{2.206}$$

and extend the definition of $\hat{\mathfrak{D}}_{\underline{m}}$ as the spatial G- and $SU(2)$-covariant derivative made from the hatted connections $\hat{A}^I$ and $\hat{\vec{B}}$ by , which also includes the affine and spin connections of the base spatial manifold.

The Eq. (2.195) is purely spatial in the time-like case and it becomes, in 4-dimensional notation[12]

$$\hat{\mathfrak{D}}_m q^X = \Phi^r{}_m{}^n\,\hat{\mathfrak{D}}_n q^Y\;J^r{}_Y{}^X\,. \tag{2.207}$$

We notice that this equation, although written in terms of covariant derivatives, impose no integrability condition on the gauge connections. That is, as equation for q^X it has always *local* solution for any given vector fields $\hat{A}^I$.

Projecting this equation along the Killing vectors k_I yields an important relation,

$$k_{I\,X}\hat{\mathfrak{D}}_m q^Y = -2\vec{\Phi}_m{}^n\hat{\mathfrak{D}}_n\vec{P}_I\,. \tag{2.208}$$

This projection is the one which appears in the Maxwell equations (2.177).

Let us study the differential equations for the two-forms $\vec{\Phi}$. The projection of Eq. (2.189) along V says that they are time-independent in the gauge (2.196):

[11] Strictly speaking the action of a 4-dimensional spatial covariant derivative on e^0 which contains dt is not well-defined. It is understood that $\hat{\mathfrak{D}}(f\,dt) = \hat{\mathfrak{D}}f\wedge dt$.

[12] From now on spatial flat indices refer to the 4-dimensional spatial metric $h_{\underline{mn}}$.

$$\partial_t \vec{\Phi}_{mn} = 0\,. \tag{2.209}$$

The components of Eq. (2.187) can be explicitly evaluated using the 5-dimensional metric Eq. (2.198) and the expression for the field strengths Eq. (2.201). Only the spatial components of the 5-dimensional covariant derivative give new information:

$$\hat{\mathfrak{D}}_m \vec{\Phi}_{np} = 0\,. \tag{2.210}$$

This is a condition for the anti-self-dual part of the spin connection ξ of the base spatial manifold. Indeed we can solve for ξ^- in an arbitrary frame and $SU(2)$ gauge:

$$\xi^-{}_{mnp} = -\hat{\vec{B}}_m \cdot \vec{\Phi}_{np} - \tfrac{1}{4}\partial_m \vec{\Phi}_{nq} \cdot \vec{\Phi}_{qp}\,, \tag{2.211}$$

where we have used the (Fierz) identity

$$\vec{\Phi}_{mn} \cdot \vec{\Phi}_{pq} = \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np} - \epsilon_{mnpq}\,. \tag{2.212}$$

The meaning of relation (2.211) becomes clearer in a frame and $SU(2)$ gauge in which the $\vec{\Phi}$s are constant: the $SU(2)$ connection $\hat{\vec{B}}$ is embedded into the anti-self-dual part of the spin connection of the base manifold. The same happenend in the ungauged case [85] and, again, this embedding requires the action of the $SU(2)$ generators in the fundamental and spinorial representation on spinors to be identical, i.e.

$$\epsilon^j\ i\vec{\sigma}_j{}^i \ = \ \tfrac{1}{4}\vec{\mathrm{J}}_{mn}\gamma^{mn}\ \epsilon^i\,, \tag{2.213}$$

and these conditions will appear as projectors

$$\Pi^{r\,\pm}{}_i{}^j \ = \ \tfrac{1}{2}\left[\ \delta\ \pm\ \tfrac{i}{4}\ \not{J}^{(r)}\sigma^{(r)}\right]_i{}^j\,, \tag{2.214}$$

acting on the Killing spinors.

It is interesting to study the integrability condition of Eq. (2.210), which is

$$\left[\tfrac{1}{4}R^-{}_{mnkl}\vec{\Phi}^{kl} + \vec{R}_{mn}(\hat{\vec{B}})\right] \times \vec{\Phi}_{pq} = 0\,, \tag{2.215}$$

where $\vec{R}_{mn}(\hat{\vec{B}})$ is the curvature of $\hat{\vec{B}}$, which is given by

$$\vec{R}_{mn}(\hat{\vec{B}}) = \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y \vec{R}_{XY}(\vec{\omega}) + \tfrac{1}{2}g\hat{F}^I_{mn}\vec{P}_I = -\tfrac{1}{4}\hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y \vec{J}_{XY} + \tfrac{1}{2}g\hat{F}^I{}_{mn}\vec{P}_I\,, \tag{2.216}$$

hence the integrability condition yields

$$R^{-}{}_{mnkl}\vec{\Phi}^{kl} - \hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y \vec{J}_{XY} + 2g\hat{F}^I{}_{mn}\vec{P}_I = 0\,. \tag{2.217}$$

We stress that this condition is equivalent to Eq. (2.211).

Now if we contract this expression with $\vec{\Phi}^{pn}$ we can compare it with Eq. (E.25) and doing so we obtain an expression involving the Ricci tensor of the spatial metric $h_{\underline{mn}}$

$$R_{mn}(h) = -\tfrac{1}{2}\hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_n q^Y g_{XY} + 2g^2 f^{-1} C^{IJK} h_I \vec{P}_J \cdot \vec{P}_K \delta_{mn} + g\hat{F}^{I+}{}_{mp}\vec{\Phi}_{pn} \cdot \vec{P}_I\,, \tag{2.218}$$

where we have used again the identity (2.212), and consequently the Ricci scalar

$$R(h) = -\tfrac{1}{2}\hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_m q^Y g_{XY} + 8g^2 f^{-1} C^{IJK} h_I \vec{P}_J \cdot \vec{P}_K\,. \tag{2.219}$$

In the ungauged case the Eq. (2.218) says that the Ricci tensor of the spatial metric $h_{\underline{mn}}$ is proportional to the induced metric

$$R_{mn}(h) = -\tfrac{1}{2}\partial_m q^X \partial_n q^Y g_{XY}\,. \tag{2.220}$$

On the other hand in the gauged case we can solve the Eq. (2.219) for f,

$$f = (8g^2 C^{IJK} h_I \vec{P}_J \cdot \vec{P}_K)/(R(h) + \tfrac{1}{2}\hat{\mathfrak{D}}_m q^X \hat{\mathfrak{D}}_m q^Y g_{XY})\,. \tag{2.221}$$

Solving the Killing spinor equations

We are now going to prove that the necessary conditions for having unbroken supersymmetry that we have derived in the previous section are also sufficient. Thus, we are going to assume that we have a configuration with a metric of the form Eq. (2.198), a non-Abelian gauge potential of the form Eq. (2.199) with a field strength of the form Eq. (2.201) satisfying Eqs. (2.202) and (2.203), and hyperscalars such that Eqs. (2.207) and (2.211) are satisfied.

Substituting these expressions in the KSE associated to the gaugino SUSY transformation rule Eq. (2.182), and expressing all terms in 4-dimensional language we get

$$f^{1/2}\left(2\,\hat{\not{\mathfrak{D}}}\phi^x - \tfrac{\sqrt{3}}{2} f^{1/2} h^x_I\ \tilde{\not{\Theta}}^{I+}\right) R^{-}\epsilon^i + 2g h^x_I \vec{P}^I \cdot \left(i\vec{\sigma}_j{}^i - \tfrac{1}{4}\ \vec{\not{\Phi}}\delta_j{}^i\right)\epsilon^j = 0\,. \tag{2.222}$$

where

$$R^{\pm} \equiv \tfrac{1}{2}\left(1 \pm \gamma^0\right) , \qquad \Pi^{r\pm}{}_j{}^i \equiv \tfrac{1}{2}\left(\delta \ \pm \ \tfrac{i}{4} \not{\Phi}^{(r)}\sigma^{(r)}\right)_j{}^i . \tag{2.223}$$

The projections

$$\vec{\Pi}^{+}{}_j{}^i\epsilon^j = 0 , \qquad R^{-}\epsilon^i = 0 , \tag{2.224}$$

are sufficient to solve it. All of them are necessary in the general case but in particular cases in which the coefficients of the projectors in the above and following equations vanish, only some of them may be necessary. The discussion is entirely analogous to that of the ungauged case [85].

Substituting now in Eq. (2.183) we get

$$f_X{}^{iA}\{f^{1/2}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i + 2\sqrt{3}gh^I k_I{}^X f_X{}^{iA}R^{-}\}\epsilon_i = 0 . \tag{2.225}$$

The last term vanishes with the second projection of Eqs. (2.224). On the other hand, from Eq. (2.207) we can derive the identity

$$f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X R^{+} = -f_X{}^{jA}\,\hat{\not{\mathfrak{D}}}q^X \sum_r \left(\Pi^{r+} - \Pi^{r-}\right)_j{}^i . \tag{2.226}$$

Acting on ϵ_i and imposing again the projections (2.224) we see that it leads to

$$f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i = -3f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i \qquad \Rightarrow \qquad f_X{}^{iA}\,\hat{\not{\mathfrak{D}}}q^X\epsilon_i = 0 . \tag{2.227}$$

Hence the hyperino KSE (2.225) is also solved.

The time component of the differential KSE (2.181)

is automatically satisfied by constant Killing spinors upon the use of the projections Eqs. (2.224).

Finally, the spatial components of the same equation take, using $R^-\epsilon^i = 0$, the form

$$\nabla_m\eta^i + \eta^j C_{m\,j}{}^i = 0 , \qquad \eta^i \equiv f^{-1/2}\epsilon^i . \tag{2.228}$$

Using the relation (2.211) and the projections, it becomes

$$\partial_m\eta^i + \tfrac{1}{16}\partial_m \not{\Phi}_j{}^i\eta^j = 0 , \tag{2.229}$$

where $\Phi_i{}^j = i\vec{\sigma}_i{}^j \cdot \vec{\Phi}$.

The solution of this equation is given in terms of the path-ordered exponential

$$\eta^i(x, x_0) = P \exp \left(-\tfrac{1}{16} \int_{x_0}^{x} dx_1^{\underline{m}} \partial_{\underline{m}} \not{\Phi}_j{}^i(x_1) \right) \eta_0^j \,, \tag{2.230}$$

where η_0^i is a constant spinor, or in a frame and $SU(2)$ gauge where $\vec{\Phi}$ is constant, it is just the constant spinor η_0^i.

The analysis of the amount of unbroken supersymmetry is identical to that of the ungauged case [85].

Supersymmetric solutions

As we discussed at the end of Section 2.5.1, the KSIs of the gauged theories have the same form as those of the ungauged ones, which are given in Ref. [85]. There it was proven that timelike supersymmetric configurations solve all the equations of motions if they solve the Maxwell equations. We are now going to impose those equations on the supersymmetric configurations. It is possible to show that the Bianchi identities imply the spatial components of the Maxwell equations for supersymmetric configurations using Eq. (2.208)

$$\mathcal{E}_I{}^m = 2C_{IJK} h^J (\star \mathfrak{D} F^K)^{0m} \,. \tag{2.231}$$

Thus we only need to impose the time component of the Maxwell equations on the supersymmetric configurations. This equation takes the form

$$\hat{\mathfrak{D}}^2 \left(h_I/f\right) - \tfrac{1}{12} C_{IJK} \hat{F}^J \cdot \hat{F}^K + \tfrac{2}{\sqrt{3}} C_{IJK} h^J \hat{F}^K \cdot G^- + 2g^2 f^{-2} g_{XY} k_I{}^X k_J{}^Y h^J = 0 \,, \tag{2.232}$$

where

$$G \equiv f d\omega \,. \tag{2.233}$$

This is the only equation that has to be solved if we have a configuration which we know is supersymmetric and admits a gauge potential. Constructing a supersymmetric configuration is, however, considerably more complex than in the ungauged or Abelian-gauged cases and it seems not possible to give an algorithm which automatically returns supersymmetric configurations. At any rate, a possible recipe to construct a supersymmetric configuration of a guven $N = 1, d = 5$ gauged supergravity theory is the following.

1. The objects that have to be chosen are

(a) The 4-dimensional spatial metric $h_{\underline{mn}}(x)$ admitting an almost complex structure $\vec{\Phi}_{mn}$. It determines the anti-selfdual part of its spin connection: $\xi^{-}{}_{mnp}$.

(b) A spatial 1-form $\omega_{\underline{m}}$.

(c) The $4n_H$ hyperscalar mappings $q^X(x)$ from the 4-dimensional spatial manifold to the quaternionic-Kähler manifold. They determine the (pullbacks of) the momentum map $\vec{P}_I$[13], $SU(2)$ connection $\vec{A}_{\underline{m}} = \partial_{\underline{m}} q^X \vec{\omega}_X$

(d) A spatial gauge potential $\hat{A}^I{}_{\underline{m}}$. It determines the spatial gauge field strength $\hat{F}^I{}_{\underline{mn}}$ and, together with the pullback of the $SU(2)$ connection $\vec{A}_{\underline{m}}$ and the momentum map, it determines the spatial $SU(2)$ connection $\hat{\vec{B}}$ whose definition we rewrite here for convenience:

$$\hat{\vec{B}} \equiv \vec{A} + \tfrac{1}{2} g \hat{A}^I \vec{P}_I \,.$$

(e) $\bar{n} = n_V + 1$ scalar functions h_I/f. They determine, upon use of the constraint $C_{IJK} h^I h^J h^L = 1$ the n_V scalars ϕ^x and the metric function f. Together with $\hat{A}^I{}_{\underline{m}}$ and $\omega_{\underline{m}}$ they give the full 5-dimensional guage potential $A^I{}_\mu$

$$A^I = -\sqrt{3} h^I e^0 + \hat{A}^I \,.$$

2. These objects now have to satisfy the following equations:

(a) Eq. (2.211) that embeds the spatial $SU(2)$ connection $\hat{\vec{B}}$ into the spin connection of the base spatial manifold.

$$\xi^{-}{}_{mnp} = -\hat{\vec{B}}_m \cdot \vec{\Phi}_{np} - \tfrac{1}{4} \partial_m \vec{\Phi}_{nq} \cdot \vec{\Phi}_{qp} \,,$$

(b) Eq. (2.207) that characterizes the hyperscalar mappings

$$\hat{\mathfrak{D}}_m q^X = \Phi^r{}_m{}^n \, \hat{\mathfrak{D}}_n q^Y \; J^r{}_Y{}^X \,.$$

(c) Eqs. (2.202) and (2.203)

$$\begin{aligned} h_I \hat{F}^{I+} &= \tfrac{2}{\sqrt{3}} (f d\omega)^+ \,, \\ \hat{F}^{I-} &= -2 g f^{-1} C^{IJK} h_J \vec{P}_K \cdot \vec{\Phi} \,. \end{aligned}$$

(d) Finally, Eq. (2.232)

$$\hat{\mathfrak{D}}^2 \left(h_I/f\right) - \tfrac{1}{12} C_{IJK} \hat{F}^J \cdot \hat{F}^K + \tfrac{2}{\sqrt{3}} C_{IJK} h^J \hat{F}^K \cdot G^- + 2 g^2 f^{-2} g_{XY} k_I{}^X k_J{}^Y h^J = 0 \,.$$

[13] If $n_H = 0$ they are constant Fayet-Iliopoulos terms as explained in footnote 3.

The null case

Denote the null Killing vector by l^μ. Following the same considerations as in Refs. [35, 80], we find that we can choose null coordinates u and v such that

$$l_\mu dx^\mu = f du\,, \qquad l^\mu \partial_\mu = \partial_{\underline{v}}\,, \tag{2.234}$$

where f may depend on u but not on v, and the metric can be put in the form

$$ds^2 = 2f du(dv + H du + \omega) - f^{-2}\gamma_{\underline{rs}} dx^r dx^s\,, \tag{2.235}$$

where $r, s, t = 1, 2, 3$ and the 3-dimensional spatial metric $\gamma_{\underline{rs}}$ may also depend on u but not on v. With this coordinates the partial gauge fixing (2.196), for $g \neq 0$, becomes just $A^I_{\underline{v}} = 0$. Eqs. (2.190) and (2.197) state that the scalars are v-independent.

In the null case Fierz identities imply that the 2-forms bilinears Φ^r are of the form

$$\Phi^r = du \wedge \Phi^r{}_s v^s\,, \qquad \Phi^r{}_t \Phi^s{}_t = \delta^{rs}\,, \tag{2.236}$$

where the 1-forms v^r are an orthogonal basis for the 3-dimensional spatial metric $\gamma_{\underline{rs}}$. Since the components $\Phi^r{}_s$ belong to $O(3)$ we can go to the frame $\Phi^r{}_s v^s$ by performing an three-dimensional Euclidean transformation. However, we must take care of that in this setting $SU(2)$ and Euclidean transformations are the same gauge symmetry[14]. Therefore the two-forms bilinear are given by

$$\Phi^r = du \wedge v^r\,. \tag{2.237}$$

In general we decompose the gauge potential as

$$A^I = A^I_{\underline{u}} du + \hat{A}^I\,, \tag{2.238}$$

where $\hat{A}$ is the spatial one-form made from the spatial components of A^I. Under a u-independent G-transformation $\hat{A}^I$ transforms as a gauge connection whereas $A^I_{\underline{u}}$ transforms homogeneously. We denote by $\hat{\mathfrak{D}}$ the spatial covariant derivative made with the three-dimensional affine and spin connections and the gauge connection $\hat{A}^I$.

Eq. (2.189) becomes

$$du \wedge \left[dv^r - \left(2\varepsilon^{rst}\hat{B}^t + \sqrt{3} g f^{-1} h^I P^s_I v^r\right) \wedge v^s\right] = 0\,. \tag{2.239}$$

[14] This holds even for the u-dependence, since the transverse three-dimensional metric depends in principle on u and hence we can perform a u-dependent Euclidean transformation

From this equation we may read the tridimensional spin connection (computed for constant u):

$$\varpi^{rs} = 2\varepsilon^{rst}\hat{B}^t - 2\sqrt{3}gf^{-1}h^I P_I^{[r}v^{s]} . \tag{2.240}$$

We also set a similar relation regarding the u-components

$$v_{[r}{}^{\underline{r}}\dot{v}_{s]\underline{r}} = -2\varepsilon_{rst}B^t_{\underline{u}} . \tag{2.241}$$

Substituting the 2-forms we found into Eq. (2.195) we arrive at

$$\hat{\mathfrak{D}}_r q^X J^r_X{}^Y = \sqrt{3}gf^{-1}h^I k_I^Y . \tag{2.242}$$

Let us now determine the vector field strengths: Eqs. (2.184) and (2.193) lead to

$$l^\mu F^I_{\mu\nu} = 0 , \tag{2.243}$$

hence we can write

$$F^I = F^I_{+r}e^+ \wedge e^r + \tfrac{1}{2}f^2 F^I_{rs}e^r \wedge e^s = F^I_{+r}du \wedge v^r + \tfrac{1}{2}F^I_{rs}v^r \wedge v^s . \tag{2.244}$$

From Eq. (2.186) we get

$$h_I F^I_{rs} = -\sqrt{3}\varepsilon_{rst}f^{-2}\partial_t f + 2gf^{-2}\varepsilon_{rst}h^I P_I^t , \qquad \partial_t \equiv v_t{}^{\underline{s}}\partial_{\underline{s}} . \tag{2.245}$$

On the other hand Eq. (2.194) yields

$$h_I^x F^I_{rs} \;=\; -\varepsilon_{rst}f^{-1}\,\hat{\mathfrak{D}}_t\phi^x + 2gf^{-2}\varepsilon_{rst}h_I^x P^I_t , \tag{2.246}$$

which, together with the previous equation and the definition of h_I^x give

$$F^I_{rs} \;=\; \sqrt{3}\varepsilon_{rst}\hat{\mathfrak{D}}_t(h^I/f) + 2gf^{-2}\varepsilon_{rst}P^{It} . \tag{2.247}$$

From the $++r$ components of Eq. (2.187) we get

$$h_I F^I_{+r} = -\tfrac{1}{2\sqrt{3}}f^2\varepsilon_{rst}F_{st} , \tag{2.248}$$

where

$$F = \hat{d}\omega . \tag{2.249}$$

The components $h_I^x F^I_{+r}$ are not determined by supersymmetry and we parametrize them by 1-forms ψ^I satisfying $h_I\psi^I = 0$. In conclusion, the vector field strengths are given by

$$F^I = (\tfrac{1}{\sqrt{3}}f^2h^I\hat{\star}F - \psi^I) \wedge du + \sqrt{3}\hat{\star}\hat{\mathfrak{D}}(h^I/f) + gf^{-2}P^{Ir}\varepsilon_{rst}v^s \wedge v^t . \tag{2.250}$$

Using the decomposition of the potential, Eq. (2.238), in the above equation we obtain the conditions

$$\hat{F}^I = \sqrt{3}\hat{\star}\hat{\mathfrak{D}}(h^I/f) + gf^{-2}P^{Ir}\varepsilon_{rst}v^s \wedge v^t\,, \tag{2.251}$$

$$\hat{\mathfrak{D}}A^I_{\underline{u}} - \partial_{\underline{u}}\hat{A}^I = \tfrac{1}{\sqrt{3}}{}^2 h^I\hat{\star}F - \psi^I\,. \tag{2.252}$$

Solving the Killing spinor equations

Seeing that we have exhausted the results flowing forth from the KSIs, it is time to have a wee look at the KSEs. Let us start by plugging our configuration into Eq. (2.182) and (2.183): They become

$$\left(f^{-1}\mathfrak{D}_{\underline{u}}\phi^x + f\hat{\mathfrak{D}}_r\phi^x\gamma^r\gamma^- + h^x_I\,\not{\psi}^I - 2gh^x_I P^{Ir}\gamma^r\gamma^-\right)\gamma^+\epsilon^i + 2gh^x_I P^{Ir}(\gamma^r + i\sigma^r)_j{}^i\epsilon^j = 0\,, \tag{2.253}$$

$$f_X{}^{iA}\left[f^{-1}\mathfrak{D}_{\underline{u}}q^X\gamma^+\epsilon_i + f\hat{\mathfrak{D}}_r q^X(\gamma^r - i\sigma^r)_i{}^j\epsilon_j\right] = 0\,. \tag{2.254}$$

As is usual in wave-like supersymmetric solutions, the $-$ component of the susy variation (2.181) is identically satisfied by an v-independent spinor which satisfy the lightlike constraint $\gamma^+\epsilon^i = 0$ and the remainder of the components simplify greatly due to this constraint: The ones in the r-directions reduce, after using Eqs. (2.245,2.248) and (2.240), to

$$\partial_r\epsilon^i + \hat{B}^s_r(\gamma^s + i\sigma^s)_j{}^i\epsilon^j = 0 \tag{2.255}$$

where in the last step we made use of Eq. (2.240). If we then introduce the projection operators (no sum over r!)

$$\Pi^r{}_i{}^j = \tfrac{1}{2}\left(\delta - i\gamma^{(r)}\sigma^{(r)}\right)_i{}^j \quad ; \quad \Pi^{r2} = \Pi^r \quad ; \quad [\,\Pi^r\,,\,\Pi^s\,] = 0\,, \tag{2.256}$$

the above equation is solved by imposing the condition $\Pi^r\epsilon = 0$, for every r for which B^r does not vanish, leading to a Killing spinor that can only depend on u.

The penultimate equation that needs to be checked is the gravitino variation in the u-direction. This calculation leads to

$$\partial_u\epsilon + (B^r_{\underline{u}} + \tfrac{1}{2\sqrt{3}}gfh^I P^r_I\gamma^-)(\gamma^r + i\sigma^r)_j{}^i\epsilon^j = 0\,, \tag{2.257}$$

where we have imposed the partial gauge fixing (2.241) and the projection $\gamma^+\epsilon^i = 0$.

A consequence of this analysis is that the Killing spinor is constant.

Equations of motion

In the null case, the KSIs contain far less restrictive information than in the timelike case, and as one can see from the KSIs in Ref. [85], there are more independent equations of motion to be checked.

The component $\mathcal{E}_I^+$ of the Maxwell equations vanishes identically. The transverse components $\mathcal{E}_I^r$ also vanish, although this is not so easy to see due to the g-terms which involve the hyperscalars. To handle with these terms one can use Eqs. (2.240) and (2.242) and also several properties of the momentum map given in appendix E. Therefore, the only non-vanishing component of the supersymmetric Maxwell equations is

$$\begin{aligned} f^{-2}\mathcal{E}_I^- &= \tfrac{\sqrt{3}}{2}\varepsilon_{rst}F_{rs}\left(f\hat{\mathfrak{D}}_t h_I + \tfrac{4}{3\sqrt{3}}gC_{IJK}h^J P^K_t\right) + \hat{\mathfrak{D}}_r(\psi_{I\,r}/f) \\ &\quad -2C_{IJK}\psi^J_r\left[\hat{\mathfrak{D}}_r(h^K/f) + \tfrac{2}{\sqrt{3}}gf^{-2}P^K_r\right] + gf^{-3}\left(k_{I\,x}\mathfrak{D}_{\underline{u}}\phi^x + k_{I\,X}\mathfrak{D}_{\underline{u}}q^X\right)\,. \end{aligned} \tag{2.258}$$

Let us now analyze the way in which the above equation together with Eqs. (2.251) and (2.252) are solved. If we define $h^I/f = K^I$, such that $f^{-3} = C_{IJK}K^IK^JK^K$, then Eq. (2.251) says that $\hat{A}^I$ and K^I are subject to

$$\hat{F}^I = \sqrt{3}\hat{\star}\hat{\mathfrak{D}}K^I + gf^{-2}P^{Ir}\varepsilon_{rst}v^s\wedge v^t\,, \tag{2.259}$$

which has the integrability condition

$$\hat{\mathfrak{D}}^2K^I + \tfrac{2}{\sqrt{3}}g\hat{\mathfrak{D}}_r(f^{-2}P^{I\,r}) = 0\,. \tag{2.260}$$

On the other hand Eq. (2.252) can be understood as a formula for ψ^I,

$$\psi^I = \tfrac{1}{\sqrt{3}}f^2h^I\hat{\star}F - \hat{\mathfrak{D}}A^I_{\underline{u}} + \partial_{\underline{u}}\hat{A}^I\,. \tag{2.261}$$

The constraint $h_I\psi^I = 0$ implies

$$\tfrac{1}{\sqrt{3}}f^2\hat{\star}F - h_I\hat{\mathfrak{D}}A^I_{\underline{u}} + h_I\partial_{\underline{u}}\hat{A}^I = 0\,. \tag{2.262}$$

Due to the relation $F=\hat{d}\omega$, the above equation is the equation that, if manage to fix the $\hat{A}^I$, will determine ω.

In the ungauged case, $\frac{1}{\sqrt{3}}\hat{A}^I$ coincides with what was called α^I in Ref. [85] and $-A^I_{\underline{u}}$ does with M^I.

Plugging Eqs. (2.260) and (2.261) into the Maxwell equations we see that

$$\begin{aligned}\tfrac{1}{2}f^{-2}\mathcal{E}_I{}^{-} \;=\;& \hat{\mathfrak{D}}^2 L_I+\tfrac{2}{\sqrt{3}}g\hat{\mathfrak{D}}_r(f^{-2}C_{IJK}P^J_r A^K_{\underline{u}})+\tfrac{1}{2}gf^{-3}\left(k_{I\,x}\mathfrak{D}_{\underline{u}}\phi^x+k_{I\,X}\mathfrak{D}_{\underline{u}}q^X\right)\\ &+C_{IJK}\left\{\hat{\mathfrak{D}}_r[K^J(\partial_{\underline{u}}\hat{A}^K)_r]-(\partial_{\underline{u}}\hat{A}^J)_r(\hat{\mathfrak{D}}_r K^K+\tfrac{2}{\sqrt{3}}gf^{-2}P^K_r)\right\}\,,\end{aligned} \tag{2.263}$$

where we have defined the combinations

$$L_I\equiv C_{IJK}K^J A^K_{\underline{u}}\,. \tag{2.264}$$

With these variables the equation for ω, Eq. (3.84), takes the form

$$\hat{\star}F=\sqrt{3}(K^I\hat{\mathfrak{D}}L_I-L_I\hat{\mathfrak{D}}K^I)-\sqrt{3}K_I\partial_{\underline{u}}\hat{A}^I\,. \tag{2.265}$$

Observe that under u-dependent G transformations the variables $\hat{A}^I$, $A^I_{\underline{u}}$, L_I and the bosonic scalars get shifts in such a way that all the above equations remains unaltered. This is not surprising because this kind of transformations are part of the symmetries of the theory. We may withdraw this symmetry by imposing the following partial gauge fixing

$$C_{IJK}\left\{\hat{\mathfrak{D}}_r[K^J(\partial_{\underline{u}}\hat{A}^K)_r]-(\partial_{\underline{u}}\hat{A}^J)_r(\hat{\mathfrak{D}}_r K^K+\tfrac{2}{\sqrt{3}}gf^{-2}P^K_r)\right\}+\tfrac{1}{2}gf^{-3}\left(k_{I\,x}\dot{\phi}^x+k_{I\,X}\dot{q}^X\right)=0 \tag{2.266}$$

such that the Maxwell equation is reduced to

$$\tfrac{1}{2}f^{-2}\mathcal{E}_I{}^{-}=\hat{\mathfrak{D}}^2L_I+\tfrac{2}{\sqrt{3}}g\hat{\mathfrak{D}}_r(f^{-2}C_{IJK}P^J_r A^K_{\underline{u}})+\tfrac{1}{2}g^2f^{-3}A^J_{\underline{u}}\left(k_{I\,x}k_J{}^x+k_{I\,X}k_J{}^X\right)\,. \tag{2.267}$$

In this approach the u-independent G transformations are keeped as part of the symmetries characterizing the solutions.

In the analysis of the Einstein equations it is useful to perform the following change of variables

$$H=-\tfrac{1}{2}L_I A^I_{\underline{u}}+N\,. \tag{2.268}$$

In general there is a gauge freedom in setting the one-form ω given in Eq. (3.84) corresponding to shifts in the coordinate v. This transformation must be accompanied with a shift in H (or N). Similarly as we proceeded with the u-dependence of the G transormations, we may fix the u-dependence of the gauge transformations of ω by demanding

$$
\begin{aligned}
&\nabla_r(\dot{\omega})_r + 3(\dot{\omega})_r\partial_r \log f = \\
&\quad -\tfrac{1}{2}f^{-3}(\ddot{\gamma})_{rr} - \tfrac{1}{4}f^{-3}(\dot{\gamma})^2 + \tfrac{3}{2}f^{-4}\dot{f}(\dot{\gamma})_{rr} + 3f^{-3}[\partial^2_{\underline{u}}\log f - 2(\partial_{\underline{u}}\log f)^2] \\
&\quad -\tfrac{1}{2}f^{-3}\left[g_{xy}(\dot{\phi}^x\dot{\phi}^y + 2g\dot{q}^x A^I_{\underline{u}}k_I{}^y) + g_{XY}(\dot{q}^X\dot{q}^Y + 2g\dot{q}^X A^I_{\underline{u}}k_I{}^Y)\right] \\
&\quad +C_{IJK}K^I\left[(\partial_{\underline{u}}\hat{A}^J)_r(\partial_{\underline{u}}\hat{A}^K)_r - 2\hat{\mathfrak{D}}_r A^J_{\underline{u}}(\partial_{\underline{u}}\hat{A}^K)_r\right].
\end{aligned}
\tag{2.269}
$$

After performing these steps, the $\mathcal{E}_{++}$ component of the Einstein equations becomes

$$
-f^{-1}\mathcal{E}_{++} = \nabla^2 N + \tfrac{1}{\sqrt{3}}g\hat{\mathfrak{D}}_r(f^{-2}C_{IJK}P^I_r A^J_{\underline{u}}A^K_{\underline{u}}) + \tfrac{1}{2}gf^{-3}A^I_{\underline{u}}A^J_{\underline{u}}(g_{xy}k_I{}^x k_J{}^y + g_{XY}k_I{}^X k_J{}^Y)\,.
\tag{2.270}
$$

3

Supersymmetric solutions of $N = 4$, $d = 4$ Supergravity

In this chapter we return to the problem of finding all the supersymmetric configurations of $N = 4, d = 4$ supergravity, partially solved by Tod in Ref. [5]. We use tensor methods, based on the bilinears of complex chiral spinors with $SU(4)$ indices, which allows us to keep manifest the S and T dualities of the theory at all stages in our analysis and in the field configurations, as it happens in the solutions studied in Ref. [13]. The formalism used here can be used as starting point for the study of more complicated theories such as gauged and matter-coupled $N = 4, d = 4$ theories and there is work in progress in these directions.

The toroidal compactification of the heterotic string effective action ($N = 1, d = 10$ supergravity coupled to 16 vector multiplets) gives ungauged $N = 4, d = 4$ supergravity coupled to 22 (matter) vector multiplets [86] and a consistent truncation of the matter vector multiplets gives the pure theory that we study here. Thus, all the solutions we will find are also solutions of the heterotic string effective action. The truncation preserves some of the $SO(6, 22; \mathbb{Z})$ T duality symmetry and the theory is invariant under the continuous group $SO(6) \sim SU(4)$ which naturally occurs as a *hidden symmetry* of the theory[1] [89]. The theory also has an S duality which manifest itself as a continuous $SL(2, \mathbb{R})$ *hidden symmetry*. It was this symmetry which lead to the S duality conjectures in the corresponding superstring theory [90]- [91]. We will also keep this symmetry manifest at all stages in our analysis.

[1]The first $N = 4, d = 4$ theory, constructed in Ref. [87] had only $SO(4)$ invariance. We will work with the $SU(4)$ theory of Ref. [88].

3.1 Results

Let us now describe our results for supersymmetric solutions, leaving the more general conditions for supersymmetric configurations which may or may not be solutions of the equations of motion.

There are two types of supersymmetric solutions in $N = 4, d = 4$ supergravity admitting at least one Killing spinor ϵ_I, that can be characterized by the causal nature of the vector bilinear $V^a = i\bar{\epsilon}^I \gamma^a \epsilon_I$, which is always a non-spacelike Killing vector.

Timelike V^a: Supersymmetric solutions are determined by a choice of 6 time-independent complex scalars M_{IJ} and a complex scalar τ that in general may depend on the spatial coordinates x, z, z^*. The M_{IJ}s have to satisfy two conditions:

1. Their matrix must have vanishing Pfaffian

$$\varepsilon^{IJKL} M_{IJ} M_{KL} = 0\,. \tag{3.1}$$

2. They must be such that the 1-form ξ defined in Eq. (3.77) takes the form

$$\xi = \pm\tfrac{i}{2}(\partial_{\underline{z}} U dz - \partial_{\underline{z}^*} U dz^*) + \tfrac{1}{2} d\lambda\,, \tag{3.2}$$

for some real functions $U(z, z^*)$ and $\lambda(x, z, z^*)$[2]. Observe that it is the function U that makes ξ non-trivial.

τ and M_{IJ} must satisfy the 3-dimensional differential equations

$$\nabla_{\underline{i}}(e^{2i\lambda} A^{\underline{i}}) - e^{2i\lambda}[\partial_{\underline{z}}(e^{-2U}) A_{\underline{z}^*} - \partial_{\underline{z}^*}(e^{-2U}) A_{\underline{z}}] = 0\,, \tag{3.3}$$

both for

$$A = \frac{d\tau}{\Im\mathrm{m}\tau |M|^2}\,, \quad \text{and} \quad A = \frac{d[(\Im\mathrm{m}\tau)^{1/2} M^{IJ}]}{\Im\mathrm{m}\tau |M|^2}\,, \qquad |M|^2 = M^{IJ} M_{IJ}\,, \tag{3.4}$$

relative to the 3-dimensional metric

$$\gamma_{\underline{ij}} dx^{\underline{i}} dx^{\underline{j}} = dx^2 + 2e^{2U(z,z^*)} dz dz^*\,, \tag{3.5}$$

whose triviality is associated to that of the connection ξ. Then, the metric is given by

[2] A general *Ansatz* that satisfies these two conditions is given in Eq. (3.211).

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}(dx^2 + 2e^{2U}dzdz^*)\,, \tag{3.6}$$

where $\omega = \omega_{\underline{i}} dx^i$ satisfies

$$f_{ij} = 4|M|^{-2}\epsilon_{ijk}\left(\xi_k - \frac{\partial_k \Re\mathrm{e}\tau}{4\Im\mathrm{m}\tau}\right)\,, \qquad f_{\underline{ij}} \equiv 2\partial_{[\underline{i}}\omega_{\underline{j}]}\,. \tag{3.7}$$

again relative to the above 3-dimensional metric and the vector field strengths are given by

$$F_{IJ} = \frac{1}{2|M|^2}\left\{\hat{V}\wedge dE_{IJ} - {}^\star\!\left[\hat{V}\wedge\left(\frac{\Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}dE_{IJ} - \frac{1}{\Im\mathrm{m}\,\tau}dB_{IJ}\right)\right]\right\}\,, \tag{3.8}$$

where

$$\begin{aligned}
\hat{V} &= \sqrt{2}|M|^2(dt+\omega)\,,\\
E_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(M_{IJ} + \tilde{M}_{IJ})\,,\\
B_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(\tau M_{IJ} + \tau^*\tilde{M}_{IJ})\,,
\end{aligned} \tag{3.9}$$

Examples of solutions corresponding to specific choices of M_{IJ} and τ are given in Section 3.4.3, but it is clear that there are two different kinds of solutions which differ by the triviality of the connection ξ and the 3-dimensional metric. The case in which ξ is trivial was completely solved by Tod in Ref. [5].

Null V^a This case (called *degenerate* by Tod) was essentially solved by Tod in Ref. [5], but we study it here again for the sake of completeness and to refine his results. There are two subcases which we call A and B and which are associated to $U(1)$ holonomy in a null direction and in a pair of spacelike directions, respectively, and describe *pp*-waves and the *stringy cosmic strings* of Ref. [92].

Case A: Each solution in this class is determined by 5 arbitrary functions of u: ϕ_I, τ. Given these functions, the metric and vector field strengths are given by

$$\begin{aligned}
ds^2 &= 2du[dv + K(u,z,z^*)du] - 2dzdz^*\,,\\
F_{IJ} &= \tfrac{1}{2}(\mathcal{F}_{IJ} + \tfrac{1}{2}\varepsilon_{IJKL}\mathcal{F}^{KL})du\wedge dz^*\,,
\end{aligned} \tag{3.10}$$

where

$$
\begin{aligned}
\mathcal{F}_{IJ} &= \frac{8\sqrt{2}}{(\Im\mathrm{m}\,\tau)^{1/2}}\dot{\phi}_{[I}\phi_{J]}\,, \\
2\partial_{\underline{z}}\partial_{\underline{z}^*}K &= \frac{|\dot{\tau}|^2}{(\Im\mathrm{m}\,\tau)^2} + \tfrac{1}{16}\Im\mathrm{m}\,\tau\,\mathcal{F}^2\,.
\end{aligned}
\tag{3.11}
$$

Case B: These are well-known solutions determined by a choice of (in this case) antiholomorphic function $\tau = \tau(z^*)$. The vector field strengths vanish[3] and the metric takes the form

$$ds^2 = 2dudv - 2e^{2U}dzdz^*\,, \qquad e^{2U} = \Im\mathrm{m}(\tau)\,. \tag{3.12}$$

As for the unbroken supersymmetries of these solutions, they all preserve generically 1/4 of the supersymmetries. It is not easy to find generic conditions for the solutions to preserve 1/2 (although this has been studied in special cases, see Ref. [13]). As for maximally supersymmetric solutions, we only expect Minkowski spacetime, since, otherwise, there would be another maximally supersymmetric solution of $N=1, d=10$ supergravity different from 10-dimensional Minkowski spacetime.

3.2 Pure, ungauged, $N=4, d=4$ supergravity

The bosonic fields of $N=4, d=4$ supergravity multiplet are:

1. The Einstein metric $g_{\mu\nu}$.

2. The complex scalar τ that parametrizes an $SL(2,\mathbb{R})/U(1)$ coset space. In terms of its real and imaginary parts (the axion a and the dilaton ϕ) it is written $\tau = a + ie^{-\phi}$.

3. The 6 $U(1)$ vector fields whose complex combinations we label with an antisymmetric pair of $SU(4)$ indices $A_{IJ\,\mu}$, $I, J = 1, \cdots, 4$ and are subject to the reality constraint

$$A_{IJ\,\mu} = \tfrac{1}{2}\varepsilon_{IJKL}A^{KL}{}_{\mu}\,, \tag{3.13}$$

[3] These solutions are given in Ref. [5] in different coordinates in which the metric functions have dependence on u, but this dependence can be eliminated.

where we rise and lower all $SU(4)$ indices by complex conjugation: $A^{IJ}{}_{\mu} \equiv (A_{IJ\,\mu})^*$. Their field strengths are $F_{IJ} = dA_{IJ}$ and are subject to the same reality constraint.

The fermionic fields of this supermultiplet, which are always 4-component (complex) Weyl spinors, are

1. The 4 dilatini χ_I, which, with lower $SU(4)$ indices, have positive chirality.
2. The 4 gravitini $\psi_{I\,\mu}$ which, with lower $SU(4)$ indices, have negative chirality.

Complex conjugation raises the $SU(4)$ indices and reverses the chiralities.

There are two global (*hidden*) symmetries in the ungauged theory: $SU(4) \sim SO(6)$, associated to stringy T dualities [93] and $SL(2,\mathbb{R})$, which is associated to a stringy S duality [90]- [91] and leaves invariant the equations of motion but not the action. $SU(4)$ acts on all the fields in the obvious way:

$$\chi^{I\,\prime} = U^I{}_J \chi^J\,, \qquad \chi_I{}' = \chi_J (U^\dagger)^J{}_I\,, \tag{3.14}$$

etc. The matrix $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts on τ via fractional-linear transformations

$$\tau' = \frac{a\tau + b}{c\tau + d}\,. \tag{3.15}$$

An alternative, linear, description of the action of $\Lambda \in SL(2,\mathbb{R})$ on τ can be made using the symmetric $SL(2,\mathbb{R})$ matrix

$$\mathcal{M} \equiv \frac{1}{\Im\mathrm{m}\,\tau} \begin{pmatrix} |\tau|^2 & \Re\mathrm{e}\,\tau \\ \Re\mathrm{e}\,\tau & 1 \end{pmatrix}\,. \tag{3.16}$$

The fractional-linear transformations of τ are equivalent to the rule

$$\mathcal{M}' = \Lambda \mathcal{M} \Lambda^T\,. \tag{3.17}$$

Observe that the matrix $S \equiv i\sigma^2$ is invariant under $SL(2,\mathbb{R})$ transformations:

$$\Lambda S \Lambda^T = S\,. \tag{3.18}$$

The action of $\Lambda \in SL(2,\mathbb{R})$ on the vector fields is best described by defining the $SL(2,\mathbb{R})$-dual $\tilde{F}_{IJ}$ of the field strength by

$$\tilde{F}_{IJ} \equiv \tau F_{IJ}{}^{+} + \bar{\tau} F_{IJ}{}^{-} = \Re\mathrm{e}\tau F_{IJ} - \Im\mathrm{m}\tau \star F_{IJ}\,. \tag{3.19}$$

Then, the pair $\tilde{F}_{IJ}$, F_{IJ} transforms as an $SL(2,\mathbb{R})$ doublet, i.e.

$$\vec{F}_{IJ} \equiv \begin{pmatrix} \tilde{F}_{IJ} \\ F_{IJ} \end{pmatrix}\,, \qquad \vec{F}'_{IJ} = \Lambda \vec{F}_{IJ}\,. \tag{3.20}$$

This implies for $F_{IJ}{}^{\pm}$

$$F'_{IJ}{}^{+} = (c\tau + d) F_{IJ}{}^{+}\,, \qquad F'_{IJ}{}^{-} = (c\bar{\tau} + d) F_{IJ}{}^{-}\,. \tag{3.21}$$

Defining the phase of $c\tau + d$ by

$$e^{2i\varphi} \equiv \frac{c\tau + d}{c\bar{\tau} + d}\,, \tag{3.22}$$

we find that, under $SL(2,\mathbb{R})$ several fields and combinations of fields get a local $U(1)$ phase

$$\begin{aligned} \chi'_I &= e^{-3i\varphi/2}\chi_I\,, & \psi'_{I\,\mu} &= e^{i\varphi/2}\psi_{I\,\mu}\,, \\ \left(\frac{\partial_\mu \tau}{\Im\mathrm{m}\,\tau}\right)' &= e^{-2i\varphi}\left(\frac{\partial_\mu \tau}{\Im\mathrm{m}\,\tau}\right)\,, & \left[\sqrt{\Im\mathrm{m}\,\tau} F_{IJ}{}^{\pm}{}_{\mu\nu}\right]' &= e^{\pm i\varphi}\left[\sqrt{\Im\mathrm{m}\,\tau} F_{IJ}{}^{\pm}{}_{\mu\nu}\right]\,, \end{aligned} \tag{3.23}$$

corresponding to $U(1)$ charges $-3, 1, -4$ and ± 2 respectively. The combination

$$Q_\mu \equiv \tfrac{1}{4}\frac{\partial_\mu \Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}\,, \tag{3.24}$$

transforms as a $U(1)$ gauge field, $Q'_\mu = Q_\mu + \frac{1}{2}\partial_\mu\varphi$ and this allows us to define a $U(1)$-covariant derivative

$$\mathcal{D}_\mu = \nabla_\mu - iqQ_\mu\,, \tag{3.25}$$

acting on fields with $U(1)$ charge q. Complex conjugation reverses chirality and these $U(1)$ charges.

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|}\left[R + \tfrac{1}{2}\frac{\partial_\mu \tau\, \partial^\mu \bar{\tau}}{(\Im\mathrm{m}\,\tau)^2} - \tfrac{1}{16}\Im\mathrm{m}\,\tau F^{IJ\,\mu\nu}F_{IJ\,\mu\nu} - \tfrac{1}{16}\Re\mathrm{e}\,\tau F^{IJ\,\mu\nu}\star F_{IJ\,\mu\nu}\right]\,. \tag{3.26}$$

It is useful to introduce the following notation for the equations of motion of the bosonic fields:

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \qquad \mathcal{E} \equiv -\frac{2\Im\mathrm{m}\tau}{\sqrt{|g|}}\frac{\delta S}{\delta \tau}\,, \qquad \mathcal{E}^{IJ\,\mu} \equiv \frac{8}{\sqrt{|g|}}\frac{\delta S}{\delta A_{IJ\,\mu}}\,. \tag{3.27}$$

Then, the equations of motion take the form

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \tfrac{1}{2}(\Im\mathrm{m}\,\tau)^{-2}[\partial_{(\mu}\tau\partial_{\nu)}\bar{\tau} - \tfrac{1}{2}g_{\mu\nu}\partial_\rho\tau\partial^\rho\bar{\tau}] - \tfrac{1}{4}\Im\mathrm{m}\,\tau F_{IJ}{}^{+}{}_{\mu}{}^{\rho}F^{IJ-}{}_{\nu\rho}, \tag{3.28}$$

$$\mathcal{E} = \mathcal{D}_\mu\left(\frac{\partial^\mu\bar{\tau}}{\Im\mathrm{m}\,\tau}\right) - \tfrac{i}{8}\Im\mathrm{m}\,\tau F^{IJ\,+\rho\sigma}F_{IJ}{}^{+}{}_{\rho\sigma}\,, \tag{3.29}$$

$$\mathcal{E}^{IJ\,\mu} = \nabla_\nu\star\tilde{F}^{IJ\,\nu\mu}\,. \tag{3.30}$$

The Maxwell equation $\mathcal{E}^{IJ\,\mu}$ transforms as an $SL(2,\mathbb{R})$ doublet together with the Bianchi identity which we denote for convenience $\mathcal{B}^{IJ\,\mu}$

$$\mathcal{B}^{IJ\,\mu} \equiv \nabla_\nu\star F^{IJ\,\nu\mu}\,. \tag{3.31}$$

It is easy to see that the combinations

$$\frac{\mathcal{E}_{IJ}{}^\mu - \bar{\tau}\mathcal{B}_{IJ}{}^\mu}{\sqrt{\Im\mathrm{m}\,\tau}}\,, \qquad \frac{\mathcal{E}_{IJ}{}^\mu - \tau\mathcal{B}_{IJ}{}^\mu}{\sqrt{\Im\mathrm{m}\,\tau}}\,, \tag{3.32}$$

have $U(1)$ charges $+2$ and -2, respectively. The equation of motion of the complex scalar $\mathcal{E}$ has $U(1)$ charge $+4$ and the Einstein equation is neutral.

For vanishing fermions, the supersymmetry transformation rules of the gravitini and dilatini, generated by 4 spinors ϵ_I of negative chirality and $U(1)$ charge $+1$, are

$$\delta_\epsilon\psi_{I\,\mu} = \mathcal{D}_\mu\epsilon_I - \tfrac{i}{2\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}F_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu\epsilon^J\,, \tag{3.33}$$

$$\delta_\epsilon\chi_I = \tfrac{1}{2\sqrt{2}}\frac{\not{\partial}\tau}{\Im\mathrm{m}\,\tau}\epsilon_I - \tfrac{1}{8}\sqrt{\Im\mathrm{m}\,\tau}\,\not{F}_{IJ}{}^{-}\epsilon^J\,. \tag{3.34}$$

We also need the supersymmetry transformation rules of the bosonic bosonic fields, which take the form

$$\delta_\epsilon e_\mu{}^a = -\tfrac{i}{4}(\bar\epsilon^I\gamma^a\psi_{I\,\mu} + \bar\epsilon_I\gamma^a\psi^I{}_\mu)\,, \tag{3.35}$$

$$\delta_\epsilon \tau = -\tfrac{i}{\sqrt{2}}\Im\mathrm{m}\tau\bar\epsilon^I\chi_I\,, \tag{3.36}$$

$$\delta_\epsilon A_{IJ\,\mu} = \frac{\sqrt{2}}{\sqrt{\Im\mathrm{m}\,\tau}}\left[\bar\epsilon_{[I}\psi_{J]\,\mu} + \tfrac{i}{\sqrt{2}}\bar\epsilon_{[I}\gamma_\mu\chi_{J]} + \tfrac{1}{2}\epsilon_{IJKL}\left(\bar\epsilon^K\psi^L{}_\mu + \tfrac{i}{\sqrt{2}}\bar\epsilon^K\gamma_\mu\chi^L\right)\right] \tag{3.37}$$

3.3 Killing Spinor Identities

Using the supersymmetry transformation rules of the bosonic fields Eqs. (3.35,3.36) and (3.37) we can derive relations between the (off-shell) equations of motion of the bosonic fields that are satisfied by any field configuration $\{e^a{}_\mu, A_{IJ\,\mu}, \tau\}$ admitting Killing spinors [6, 71]. These KSIs take, for this theory, the form

$$i\bar\epsilon^I\gamma^a\mathcal{E}_a{}^\mu + \frac{1}{\sqrt{2\Im\mathrm{m}\tau}}\bar\epsilon_J\mathcal{E}^{\mu JI} = 0\,, \tag{3.38}$$

$$\bar\epsilon^I\mathcal{E} + \frac{1}{\sqrt{2\Im\mathrm{m}\tau}}\bar\epsilon_J\,\not{\mathcal{E}}^{JI} = 0\,. \tag{3.39}$$

Observe that it is implicitly assumed that the Bianchi identities are identically satisfied, i.e.

$$\mathcal{B}_{IJ}{}^\mu = 0\,, \tag{3.40}$$

and, therefore, these identities are not $SL(2,\mathbb{R})$-covariant. We may have to take this point into account when comparing with the equations that we will actually find, but we can also find (with considerably more effort) the $SL(2,\mathbb{R})$-covariant relations between the equations of motion from the integrability conditions of the Killing spinor equations (3.51) and (3.52).

Thus, acting with $\mathcal{D}_\mu$ on the Eq. (3.51) using both Eq. (3.51) and Eq. (3.52) and antisymmetrizing on the vector indices we get

$$\begin{aligned}
\mathcal{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} &= \tfrac{1}{8}\frac{\partial_{[\mu}\tau\partial_{\nu]}\bar{\tau}}{(\Im\mathrm{m}\,\tau)^2}\epsilon_I \\
&\quad -\tfrac{1}{8}\left\{R_{\mu\nu}{}^{ab}\delta_I{}^K - \Im\mathrm{m}\,\tau F_{IJ}{}^+{}_{[\mu}{}^a F^{KJ\,-}{}_{\nu]}{}^b\right\}\gamma_{ab}\epsilon_K \\
&\quad +\tfrac{1}{4\sqrt{2}}(\Im\mathrm{m}\,\tau)^{-1/2}\left\{F_{IJ}{}^+{}_{\rho[\nu}\partial_{\mu]}\tau - 2i\Im\mathrm{m}\,\tau\nabla_{[\mu|}F_{IJ}{}^+{}_{|\nu]}\right\}\gamma^\rho\epsilon^J \\
&= 0\,.
\end{aligned} \tag{3.41}$$

To extract from this integrability condition a relation between the equations of motion we act with γ^ν from the left. We get

$$4\gamma^\nu\mathcal{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} = (\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_\sigma{}^\sigma)\gamma^\nu\epsilon_I - \frac{i}{2\sqrt{2\Im\mathrm{m}\tau}}(\not{\mathcal{E}}_{IJ} - \bar{\tau}\,\not{\mathcal{B}}_{IJ})\gamma_\mu\epsilon^J = 0\,. \tag{3.42}$$

Acting now with γ^μ and using the result to eliminate $\mathcal{E}_\sigma{}^\sigma$ we get, finally the $SL(2,\mathbb{R})$-covariantization of the KSIs Eq. (3.38)

$$\mathcal{E}^\mu{}_a\gamma^a\epsilon_I - \frac{i}{\sqrt{2\Im\mathrm{m}\tau}}(\mathcal{E}_{IJ}{}^\mu - \bar{\tau}\mathcal{B}_{IJ}{}^\mu)\epsilon^J = 0\,. \tag{3.43}$$

Similarly, the $SL(2,\mathbb{R})$-covariantization of the KSIs Eq. (3.38) can be obtained by calculating $2\sqrt{2}\,\not{\mathcal{D}}\delta_\epsilon\chi_I = 0$ and takes the form

$$\mathcal{E}^*\epsilon_I - \frac{1}{\sqrt{2\Im\mathrm{m}\tau}}(\not{\mathcal{E}}_{IJ} - \tau\,\not{\mathcal{B}}_{IJ})\epsilon^J = 0\,. \tag{3.44}$$

These two identities are now manifestly $SL(2,\mathbb{R})$-covariant[4]. The comparison with our results will be easier if we multiply these equations by gamma matrices and conjugate spinors $\bar{\epsilon}_K$ and $\bar{\epsilon}^K$ from the left, to derive relations involving spinor bilinears. In the case in which the vector V^a is timelike, we get

$$\mathcal{E}^{ab} - \tfrac{1}{2}\Im\mathrm{m}\,\mathcal{E}V^aV^b - \frac{1}{\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}\Im\mathrm{m}\,(M^{IJ}\mathcal{B}_{IJ}{}^a)V^b \;=\; 0\,, \tag{3.45}$$

$$\mathcal{E}^*V^a - \frac{i}{\sqrt{2\Im\mathrm{m}\tau}}M^{IJ}(\mathcal{E}_{IJ}{}^a - \tau\mathcal{B}_{IJ}{}^a) \;=\; 0\,, \tag{3.46}$$

$$\Im\mathrm{m}[M_{IJ}(\mathcal{E}_{IJ}{}^a - \bar{\tau}\mathcal{B}_{IJ}{}^a)] \;=\; 0\,. \tag{3.47}$$

[4]See the paragraph after Eq. (3.32).

Observe that the first equation implies the off-shell vanishing of all the Einstein equations with one or two spacelike components. Further, the Einstein equation is automatically satisfied when the Maxwell, Bianchi and complex scalar equations are satisfied.

When V^a is null (we denote it by l^a), all the spinors ϵ_I are proportional and we can use the parametrization of Eq. (B.39) in Eqs. (3.43) and (3.44). Contracting with ϕ^I using the normalization Eq. (B.40) and with the conjugate spinors $\bar{\epsilon}, \bar{\epsilon^*}, \bar{\eta}, \bar{\eta^*}$, where η is an auxiliary spinor with normalization Eq. (B.46), we arrive at the identities

$$(\mathcal{E}^{\mu}{}_{a} - \tfrac{1}{2}e_{a}{}^{\mu}\mathcal{E}^{\rho}{}_{\rho})\, l^{a} = (\mathcal{E}^{\mu}{}_{a} - \tfrac{1}{2}e_{a}{}^{\mu}\mathcal{E}^{\rho}{}_{\rho})\, m^{a} = 0\,, \tag{3.48}$$

$$\mathcal{E} = 0\,, \tag{3.49}$$

$$(\mathcal{E}_{IJ}{}^{\mu} - \bar{\tau}\mathcal{B}_{IJ}{}^{\mu})\phi^{J} = 0\,. \tag{3.50}$$

where the null complex vectors are defined in Eq. (B.47). Observe that in this case supersymmetry implies that the scalar equations of motion must be automatically satisfied.

3.4 Supersymmetric configurations and solutions

3.4.1 General setup and first results

Our goal is to find all the purely bosonic field configurations of $N=4, d=4$ supergravity $\{g_{\mu\nu}, A_{IJ\,\mu}, \tau, \psi_{I\,\mu}=0, \chi_I=0\}$ which are supersymmetric, i.e. invariant under, at least, one supersymmetry transformation generated by a supersymmetry parameter $\epsilon_I(x)$. Since the supersymmetry variations of the bosonic fields are odd in fermion fields, these transformations will always vanish, but the supersymmetry variations of the fermions, for vanishing fermions, Eqs. (3.33), may only vanish for special supersymmetry parameters $\epsilon_I(x)$ (*Killing spinors*) that solve the *Killing spinor equations* (KSEs)

$$\delta_\epsilon \psi_{I\,\mu} = \mathcal{D}_\mu \epsilon_I - \tfrac{i}{2\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}\, F_{IJ}{}^{+}{}_{\mu\nu}\gamma^{\nu}\epsilon^{J} = 0\,, \tag{3.51}$$

$$2\sqrt{2}\,\delta_\epsilon \chi_I = \frac{\partial\!\!\!/\tau}{\Im\mathrm{m}\,\tau}\epsilon_I - \tfrac{1}{2\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}\; F\!\!\!/_{IJ}{}^{-}\epsilon^{J} = 0\,. \tag{3.52}$$

For a known bosonic field configuration these are, respectively differential and algebraic equations for the Killing spinor, which may or may not exist. We want to find precisely for which bosonic field configurations these equations do have at least one solution ϵ_I. Our procedure will consist in assuming the existence of such a solution and derive consistency conditions for the field configurations.

We shall be talking most of the time about supersymmetric *field configurations*. These may or may not be solutions of the classical equations of motion. There are several conceptual and practical advantages in doing so. First of all, we would like to emphasize the fact that supersymmetry does not imply by itself that the equations of motion are solved, although in general it considerably simplifies the task of solving them. Secondly, it is sometimes useful to consider that there are external sources for the fields, out of the regions in which we are solving the equations of motion. Including those regions with sources implies staying off-shell. Finally, the off-shell equations of motion of theories with gauge symmetries obey certain gauge identities. In theories with local supersymmetry and for field configurations admitting Killing spinors, the gauge identities are known as *Killing spinor identities* (KSIs) [6, 71] and can be used either to reduce the number of equations to be explicitly checked or, having at hands all the off-shell equations of motion of certain field configuration as we will, they can be used as a consistency check that it is a supersymmetric field configuration.

Since these identities are the first consistency conditions that can be derived from the KSEs, we are going to derive them in the next section. We are also going to see that they are related to the integrability conditions of the KSEs.

The procedure we will follow to find the field configurations for which the KSEs admit at least one solution will be the following:

1. We are going to reexpress the KSEs as differential and algebraic equations for the bilinears (scalars, vectors and 2-forms, see Appendix B.1.2) built with the Killing spinors.

2. We are going to find, among the bilinears, a Killing vector V^μ and decompose the vector field strengths w.r.t. to it computing $V^\rho F_{IJ}{}^+{}_{\mu\rho}$ or $V^\rho F_{IJ}{}^-{}_{\mu\rho}$ in terms of the scalar bilinears and τ and then using, Eqs. (A.20) if V is timelike and Eqs. (A.29) if V is null. These two cases have to be studied separately. One of the reasons is that, in the null case, the field strength is not completely determined by its contractions with V, but there are more differences that we are going to explain shortly and require a completely separate analysis.

3. In the timelike case, studied in Section 3.4.2 we will

   (a) Substitute the expressions of the field strengths in the algebraic KSEs ($\delta_\epsilon \chi_I = 0$) to check that it is completely solved.

(b) Substitute into the equations of motion and we will check whether the KSIs Eqs. (3.45,3.46,3.47) are indeed satisfied or there are additional conditions to be imposed. This is done in two steps: first we substitute into the equations of motion of the vector fields and the complex scalar which we have already expressed in terms of the bilinears and then, after we specify the form of the metric in terms of the bilinears, we substitute into the Einstein equations in Section 3.4.2.

(c) Substitute, finally, into the differential KSEs ($\delta_\epsilon \psi_{I\,\mu} = 0$) to solve it finding additional conditions on the bilinears and the form of the Killing spinors.

The timelike case will be completely solved by then and we will study some examples.

In the null case, which was completely solved by Tod,

(a) As explained in Appendix B.1.2 all the spinors ϵ_I are proportional $\epsilon_I = \phi_I \epsilon$ and we use first this information in the KSEs to obtain separate equations for the coefficients ϕ_I and the spinor ϵ. This requires the introduction of a $U(1)$ connection ζ that covariantizes the equations with respect to (opposite) local changes of phase of ϕ_I and ϵ.

(b) All the vectors bilinears are also proportional to the Killing vector V^a which we rename here l^a. It is convenient to introduce an auxiliary spinor to build independent vector bilinears that constitute a null tetrad. The KSEs only give partial information about the derivatives of these vectors, except for l^a, which is built with ϵ and is always covariantly constant, the very definition of a *pp*-wave space [94, 95].

(c) Although the vector field strengths and the derivatives of the vector bilinears are not completely determined, it is possible to extract information constructing the equations of motion and imposing the KSI. In particular we find that the $U(1)$ connection ζ is trivial.

(d) There are two different cases to be considered (A and B) which are essentially solved by solving first the integrability constraints.

We start with the equations $\delta_\epsilon \chi_I = 0$. We just have to multiply the from the right with gamma matrices and Dirac conjugates of Killing spinors. We have, in particular, from $\bar{\epsilon}^K \delta_\epsilon \chi_I = 0$

$$V^K{}_I \cdot \partial\tau - \tfrac{i}{2\sqrt{2}}(\Im\mathrm{m}\,\tau)^{3/2} F_{IJ}{}^- \cdot \Phi^{KJ} = 0\,, \tag{3.53}$$

and, from $\bar{\epsilon}^K \gamma_\rho \delta_\epsilon \chi_I = 0$

$$F_{IJ}{}^{-}{}_{\rho\sigma}V^{J}{}_{K}{}^{\sigma}+\tfrac{i}{\sqrt{2}}(\Im\mathrm{m}\,\tau)^{-3/2}\left(M_{IK}\partial_{\rho}\tau-\Phi_{IK\,\rho}{}^{\mu}\partial_{\mu}\tau\right)=0\,. \qquad (3.54)$$

It is possible to derive more Killing equations for the bilinears from the dilatini supersymmetry rule, but it will not be necessary.

Let us turn to the gravitini supersymmetry rules. Now we apply $SL(2,\mathbb{R})$-covariant derivative on the bilinears and use $\delta_{\epsilon}\psi_{I\,\mu}=0$ to reexpress $\mathcal{D}_{\mu}\epsilon_{I}$. We get

$$\mathcal{D}_{\mu}M_{IJ} = \tfrac{1}{\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}F_{K[I|}{}^{+}{}_{\mu\nu}V^{K}{}_{|J]}{}^{\nu}\,, \qquad (3.55)$$

$$\begin{aligned}\mathcal{D}_{\mu}V^{I}{}_{J\,\nu} &= -\tfrac{1}{2\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}\left[M_{KJ}F^{KI\,-}{}_{\mu\nu}+M^{IK}F_{JK}{}^{+}{}_{\mu\nu}\right.\\ &\quad\left.-\Phi_{KJ\,(\mu}{}^{\rho}F^{KI\,-}{}_{\nu)\rho}-\Phi^{IK}{}_{(\mu|}{}^{\rho}F_{KI}{}^{+}{}_{|\nu)\rho}\right]\,, \end{aligned} \qquad (3.56)$$

$$\begin{aligned}\mathcal{D}_{\mu}\Phi_{IJ\,\mu\nu} &= -\tfrac{1}{2\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}\left[2g_{\mu[\nu|}F_{KI}{}^{+}{}_{|\rho]\alpha}V^{K}{}_{J}{}^{\alpha}+2F_{KI}{}^{+}{}_{\nu\rho}V^{K}{}_{J\,\mu}\right.\\ &\quad\left.-3F_{KI}{}^{+}{}_{[\mu\nu|}V^{K}{}_{J\,|\rho]}+(I\leftrightarrow J)\right]\,. \end{aligned} \qquad (3.57)$$

Contracting the free indices in Eqs. (3.56) and (3.53) it is immediate to see that $V^{\mu}\equiv V^{I}{}_{I}{}^{\mu}$ is a (non-spacelike, Eq. (B.28)) Killing vector and

$$V^{\mu}\partial_{\mu}\tau=0\,. \qquad (3.58)$$

It is also immediate to prove that

$$\nabla_{\mu}V^{I}{}_{J}{}^{\mu}=0\,. \qquad (3.59)$$

Let us now consider the implications of the reality constraint of the vector field strengths on the contraction $F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu}$:

$$F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu}=\tfrac{1}{2}\varepsilon_{KIML}(F_{ML}{}^{-}{}_{\mu\nu})^{*}V^{K}{}_{J}{}^{\nu}\,. \qquad (3.60)$$

Taking the $SU(4)$ dual in both sides of this equation and taking into account the reality properties of the vectors $V^{K}{}_{J}{}^{\nu}$, we get

$$\tfrac{1}{2}\varepsilon^{SRIJ}F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu}=-\tfrac{1}{2}\left[F_{SR}{}^{-}{}_{\mu\nu}V^{\nu}+2F_{J[S|}{}^{-}{}_{\mu\nu}V^{J}{}_{|R]}{}^{\nu}\right]^{*}\,, \qquad (3.61)$$

from which we get

$$F_{SR}{}^{-}{}_{\mu\nu}V^{\nu} = -2F_{J[S|}{}^{-}{}_{\mu\nu}V^{J}{}_{|R]}{}^{\nu} - \left[\varepsilon^{SRIJ}F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu}\right]^{*}\,. \tag{3.62}$$

The first and second terms in the r.h.s. of this equation can be rewritten in terms of scalars using the antisymmetric part of Eq. (3.54) and the complex conjugate of Eq. (3.55). We get, at last,

$$F_{SR}{}^{-}{}_{\mu\nu}V^{\nu} = -\frac{\sqrt{2}i}{(\Im\mathrm{m}\,\tau)^{3/2}}M_{SR}\partial_{\mu}\tau - \frac{\sqrt{2}}{\sqrt{\Im\mathrm{m}\,\tau}}\varepsilon_{SRIJ}\mathcal{D}_{\mu}M^{IJ}\,. \tag{3.63}$$

The complex conjugate of this equation gives us $F^{SR+}{}_{\mu\nu}V^{\nu}$ and, taking the $SU(4)$-dual we get $F_{IJ}{}^{+}{}_{\mu\nu}V^{\nu}$ etc.

From this equation, contracting the free index with V^{μ} and using Eq. (3.58) we get immediately

$$V^{\mu}\partial_{\mu}M_{IJ} = 0\,. \tag{3.64}$$

Now, the use that we make of this result and the subsequent analysis will depend on the causal nature if the non-spacelike vector V^{μ}. We must distinguish between two cases: the case in which it is timelike, which we consider in section 3.4.2 and the case in which it is null (and we rename it l^{μ}), which we consider in section 3.4.4.

3.4.2 The timelike case

The vector field strengths

If $V^2 = 2M^{IJ}M_{IJ} \equiv 2|M|^2 \neq 0$ we can use Eq. (3.63) to express $F_{IJ}{}^{-}$ entirely in terms of scalars, their derivatives, and V_{μ} using Eq. (A.20):

$$F_{SR}{}^{-} = -\frac{1}{\sqrt{2}|M|^2\sqrt{\Im\mathrm{m}\,\tau}}\left\{\left[i\frac{M_{SR}}{(\Im\mathrm{m}\,\tau)}d\tau + \varepsilon_{SRIJ}\mathcal{D}M^{IJ}\right]\wedge\hat{V} - i\star[\cdots]\right\}\,. \tag{3.65}$$

Here we have added a hat to V to denote the differential form $\hat{V} \equiv V_{\mu}dx^{\mu}$ and distinguish its norm.

To solve the equations of motion it is convenient to have directly F_{IJ} and its $SL(2,\mathbb{R})$-dual $\tilde{F}_{IJ}$. Their expressions are, actually, somewhat simpler due to the following property: if $dF = 0$ (which is the equation satisfied by F_{IJ} and $\tilde{F}_{IJ}$) and $\pounds_{V}F = 0$ then $\nabla_{[\mu}(F_{\nu]\rho}V^{\rho}) = 0$ and, locally, $F_{\nu\rho}V^{\rho} = \nabla_{\nu}E$ for some scalar potential E. Thus, following Tod [5], we define

$$\nabla_\mu E_{IJ} \equiv V^\nu F_{IJ\,\nu\mu}\,, \qquad \nabla_\mu B_{IJ} \equiv V^\nu \tilde{F}_{IJ\,\nu\mu}\,, \tag{3.66}$$

and, using the above form of $F_{IJ}{}^-$ Eq. (3.65) we find

$$\begin{aligned} E_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(M_{IJ}+\tilde{M}_{IJ})\,,\\ B_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(\tau M_{IJ}+\bar{\tau}\tilde{M}_{IJ})\,, \end{aligned} \tag{3.67}$$

where

$$\tilde{F}_{IJ} = V^{-2}\left\{\hat{V}\wedge dB_{IJ} + \star\!\left[\hat{V}\wedge\left(\frac{\Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}dB_{IJ} - \frac{|\tau|^2}{\Im\mathrm{m}\,\tau}dE_{IJ}\right)\right]\right\}\,, \tag{3.68}$$

$$F_{IJ} = V^{-2}\left\{\hat{V}\wedge dE_{IJ} - \star\!\left[\hat{V}\wedge\left(\frac{\Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}dE_{IJ} - \frac{1}{\Im\mathrm{m}\,\tau}dB_{IJ}\right)\right]\right\}\,. \tag{3.69}$$

It is worth spending a moment in checking the consistency of these results. By definition, B_{IJ} and E_{IJ} must transform under $SL(2,\mathbb{R})$ as $\tilde{F}_{IJ}$ and F_{IJ}, i.e. as a doublet:

$$\vec{E}_{IJ} \equiv \begin{pmatrix} B_{IJ}\\ E_{IJ}\end{pmatrix}\,, \qquad \vec{E}'_{IJ} = \Lambda\vec{E}_{IJ}\,. \tag{3.70}$$

We can check that this is consistent with Eqs. (3.68) and (3.69) by rewriting the last two equations in the manifestly $SL(2,\mathbb{R})$-covariant form

$$\vec{F}_{IJ} = V^{-2}\left\{\hat{V}\wedge d\vec{E}_{IJ} - \star\!\left[\hat{V}\wedge(\mathcal{M}Sd\vec{E}_{IJ})\right]\right\}\,, \tag{3.71}$$

on account of Eqs. (3.16,3.17) and (3.18).

On the other hand, it is easy to check that the fact that $\vec{E}_{IJ}$ transforms as a doublet is consistent with the transformations rules of τ and M_{IJ} alone and Eqs. (3.67).

The five-dimensional metric

We define a time coordinate by

$$V^\mu\partial_\mu \equiv \sqrt{2}\partial_t\,, \tag{3.72}$$

and the metric takes the "conformastationary" form

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}\gamma_{\underline{ij}}dx^i dx^j\,, \qquad i,j=1,2,3\,, \tag{3.73}$$

where $\omega = \omega_{\underline{i}} dx^i$ is a time-independent 1-form and $\gamma_{\underline{ij}}$ is a time-independent (positive-definite!) metric on constant t hypersurfaces[5].

From Eq. (3.56) we find that V satisfies the equation

$$d\hat{V} = -\tfrac{1}{\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}[M^{IJ}F_{IJ}{}^{+} + M_{IJ}F^{IJ\,-}]\,. \tag{3.74}$$

Since

$$M^{IJ}F_{IJ}{}^{+} = -\frac{\sqrt{2}M^{IJ}}{\sqrt{\Im\mathrm{m}\,\tau}|M|^2}[\mathcal{D}M_{IJ}\wedge\hat{V} + i\star(\mathcal{D}M_{IJ}\wedge\hat{V})]\,, \tag{3.75}$$

we get

$$d\hat{V} = \frac{1}{|M|^2}\left\{d|M|^2\wedge\hat{V} + i\star\Big[(M^{IJ}\mathcal{D}M_{IJ} - M_{IJ}\mathcal{D}M^{IJ})\wedge\hat{V}\Big]\right\}\,. \tag{3.76}$$

It is also convenient to define the 1-form ξ and the 2-form Ω

$$\xi \equiv \tfrac{i}{4}|M|^{-2}(M_{IJ}dM^{IJ} - M^{IJ}dM_{IJ})\,, \tag{3.77}$$

$$\Omega \equiv 2|M|^{-2}\star\Big[(Q-\xi)\wedge\hat{V}\Big]\,. \tag{3.78}$$

ξ transforms under $SL(2,\mathbb{R})$ as

$$\xi' = \xi + \tfrac{1}{2}d\varphi\,, \tag{3.79}$$

i.e. as the $U(1)$ connection Q, which makes Ω invariant. The connection ξ is also orthogonal to V and invariant under local rescalings of the scalar matrix M_{IJ}:

$$\xi(\Lambda(x)M_{IJ}) = \xi(M_{IJ})\,, \tag{3.80}$$

a property that we will exploit later on. Further, using Eq. (B.37) we can write the curvature of this connection in the form

$$d\xi = -\tfrac{i}{2}d\frac{M^{IJ}}{|M|}\wedge d\frac{M_{KL}}{|M|}[\delta_{IJ}{}^{KL} - \mathcal{J}^{K}{}_{[I}\mathcal{J}^{L}{}_{J]}]\,, \tag{3.81}$$

that relates the triviality of ξ with the constancy of the projection $\mathcal{J}^{I}{}_{J}$.

[5]The components of the connection and curvature of this metric can be found in Appendix C.1.

Using the expressions that we have found for the Maxwell fields and their $SL(2,\mathbb{R})$ duals and using the above equation for dV rewritten in the form

$$d\hat{V} = \frac{d|M|^2}{|M|^2} \wedge \hat{V} + 2|M|^2\Omega\,, \tag{3.82}$$

As usual, the equation for ω can be derived by comparing Eq. (3.82) for the 1-form $\hat{V}$, with the exterior derivative of the expression for $\hat{V}$ in the coordinates chosen

$$\hat{V} = \sqrt{2}|M|^2(dt+\omega)\,. \tag{3.83}$$

The result is the equation

$$d\omega = \tfrac{1}{\sqrt{2}}\Omega = \tfrac{i}{2\sqrt{2}}|M|^{-4}\star\Big[(M^{IJ}\mathcal{D}M_{IJ} - M_{IJ}\mathcal{D}M^{IJ})\wedge\hat{V}\Big]\,. \tag{3.84}$$

The four-dimensional spin connection

Now we analyze the implications of the supersymmetry. From (3.133) we obatin that the $V_A{}^B$ are covariantly constant

$$\nabla_i V_{jA}{}^B - \mathcal{A}_{iA}{}^C V_{jC}{}^B + \mathcal{A}_{iC}{}^B V_{jA}{}^C = 0 \tag{3.85}$$

And similiarly we obtain the covariant constancy of V^x

$$\nabla_i V_j{}^x - \epsilon^{xyz}\mathcal{A}_{i\,y}V_{j\,z} = 0 \tag{3.86}$$

(in this eq. ∇ does not contain the spin connection) where $\mathcal{A}^x$ is the adjoint version of the $SU(2)$ connection

$$\mathcal{A}^x = i\sigma^{x}_{A}{}^{B}\mathcal{A}_B{}^A\,, \tag{3.87}$$

$$\mathcal{A}_A{}^B = -\frac{i}{2}\sigma^{x}_{A}{}^{B}\mathcal{A}_x\,. \tag{3.88}$$

The equation (3.86) can be interpreted as the vielbein postulate $\nabla_i V_j{}^x = 0$ with the $SU(2)$ connection $\mathcal{A}^x$ playing the role of the spin connection

$$o^{xy} = \epsilon^{xyz}\mathcal{A}_z\,. \tag{3.89}$$

This relation is fundamental for our purposes of solving the KSEs.

Solving the Killing spinor equations

From the equations of the bilinears we have obtained the supersymmetric bosonic fields. They consist of arbitrary axidilaton τ and

$$ds^2 = |M|^2 (dt+\omega)^2 - |M|^{-2} h_{ij} dx^i dx^j \tag{3.90}$$

$$f_{ij} = 2\partial_{[i}\omega_{j]} = 4|M|^{-2}\epsilon_{ijk}\left(\xi - Q\right)^k \tag{3.91}$$

$$\sqrt{\Im\mathrm{m}\tau} F^{+}_{IJ\,\alpha\beta} = 4\sqrt{2}|M|^{-2} f^{+}_{\alpha\beta}{}^{\mu\nu} V_\mu \left(DM_{IJ} - \frac{i}{2}\tilde{M}_{IJ}\frac{d\bar{\tau}}{\Im\mathrm{m}\tau}\right)_\nu \tag{3.92}$$

$$\sqrt{\Im\mathrm{m}\tau} F^{-}_{IJ\,\alpha\beta} = 4\sqrt{2}|M|^{-2} f^{-}_{\alpha\beta}{}^{\mu\nu} V_\mu \left(D\tilde{M}_{IJ} + \frac{i}{2}M_{IJ}\frac{d\tau}{\Im\mathrm{m}\tau}\right)_\nu \tag{3.93}$$

where V is the time and

$$f^{\pm}_{\alpha\beta}{}^{\mu\nu} = \frac{1}{2}\left(\delta_{\alpha\beta}{}^{\mu\nu} \pm \frac{i}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}\right) \tag{3.94}$$

are the complex self-dual and anti-self-dual projectors over two-forms.

The fields are independent of time

$$\partial_t \tau = \partial_t M_{IJ} = \partial_t h_{ij} = 0\ . \tag{3.95}$$

The only constraint on the scalars M_{IJ} is that the correspondig 4×4 skew-symmetric matrix is singular, this is a condition on the Pfaffian

$$\epsilon^{IJKL} M_{IJ} M_{KL} = 0\ . \tag{3.96}$$

This constraint can be stated in a way that can be more useful for operations:

$$M_{[IJ} M_{K]L} = 0\ . \tag{3.97}$$

Therefore only five of the six M_{IJ} are independent.

Let $\mathcal{J}$ be the projector

$$\mathcal{J}_I{}^J = \frac{2}{|M|^2} M_{IK} M^{JK}\ . \tag{3.98}$$

$\mathcal{J}$ is indeed a projector due to (3.97) and also the M_{IJ} are projected to $\mathcal{J}$,

$$M_{IJ} = \mathcal{J}_I{}^K M_{KJ}\ . \tag{3.99}$$

We also introduce the complementary projector of $\mathcal{J}$,

$$\tilde{\mathcal{J}}_I{}^J = \frac{2}{|M|^2}\tilde{M}_{IK}\tilde{M}^{JK} = \delta_I{}^J - \mathcal{J}_I{}^J\ . \tag{3.100}$$

The most simple solution of the constraint (3.97) with non-constant $\mathcal{J}$ is an array of only two M_{IJ} non-zero and arbitrary.

We want to analyze the KSEs with the supersymmetric expressions for the fields given above. The KSEs are

$$D_\mu \epsilon_I - \frac{i}{2\sqrt{2}}\sqrt{\Im\mathrm{m}\tau}F^+_{IJ\,\mu\nu}\gamma^\nu\epsilon^J = 0 \tag{3.101}$$

$$\frac{\partial\!\!\!/\tau}{\Im\mathrm{m}\tau}\epsilon_I - \frac{1}{2\sqrt{2}}\sqrt{\Im\mathrm{m}\tau}F\!\!\!/^-_{IJ}\epsilon^J = 0\ . \tag{3.102}$$

Setting $V^0 = \sqrt{2}|M|$, $V^i = 0$ we obtain the KSEs

$$\nabla_0\epsilon_I - \frac{i}{\sqrt{2}|M|}\gamma^k\left(DM_{IJ} - \frac{i}{2}\tilde{M}_{IJ}\frac{d\bar{\tau}}{\Im\mathrm{m}\tau}\right)_k\epsilon^J = 0 \tag{3.103}$$

$$D_i\epsilon_I + \frac{i}{\sqrt{2}|M|}\gamma^k\left(DM_{IJ} - \frac{i}{2}\tilde{M}_{IJ}\frac{d\bar{\tau}}{\Im\mathrm{m}\tau}\right)_k\gamma_i\gamma^0\epsilon^J = 0 \tag{3.104}$$

$$\gamma^k\left[\frac{d\tau}{\Im\mathrm{m}\tau}\epsilon_I + \frac{2\sqrt{2}}{|M|}\left(D\tilde{M}_{IJ} + \frac{i}{2}M_{IJ}\frac{d\tau}{\Im\mathrm{m}\tau}\right)\gamma^0\epsilon^J\right]_k = 0\ . \tag{3.105}$$

Using the spin connection we obtain the covariant derivatives

$$\nabla_0\epsilon_I = \partial_0\epsilon_I + \frac{1}{2|M|^2}M^{KL}D_kM_{KL}\gamma^{0k}\epsilon_I\ , \tag{3.106}$$

$$D_i\epsilon_I = |M|\nabla_i\epsilon_I - \frac{1}{2|M|^2}M^{KL}\partial_kM_{KL}\gamma_i{}^k\epsilon_I - i\gamma^k\gamma_iQ_k\epsilon_I\ . \tag{3.107}$$

With the help of the identity

$$dM_{IJ} = |M|^{-2}M^{KL}dM_{KL}M_{IJ} + 2\tilde{\mathcal{J}}_{[I}{}^K dM_{K|J]} \tag{3.108}$$

we may write the KSEs as

$$\begin{aligned}
&\partial_0\epsilon_I + \frac{1}{2|M|^2}\gamma^{0k}M^{KL}D_kM_{KL}\left(\epsilon_I + i\sqrt{2}\gamma^0\frac{M_{IJ}}{|M|}\epsilon^J\right)\\
&\qquad\qquad -\frac{i\sqrt{2}}{|M|}\gamma^k\left(\tilde{\mathcal{J}}_{[I}{}^K\partial_kM_{K|J]} - \frac{i}{4}\frac{\partial_k\bar{\tau}}{\Im\mathrm{m}\tau}\tilde{M}_{IJ}\right)\epsilon^J = 0
\end{aligned} \tag{3.109}$$

$$\begin{aligned}
&|M|\nabla_i\epsilon_I + \frac{i}{\sqrt{2}|M|^3}M^{KL}\partial_iM_{KL}\gamma^0M_{IJ}\epsilon^J\\
&\quad -\frac{1}{2|M|^2}\left(M^{KL}\partial_kM_{KL}\gamma_i{}^k + 2iQ_k\gamma^k\gamma_i\right)\left(\epsilon_I + i\sqrt{2}\gamma^0\frac{M_{IJ}}{|M|}\epsilon^J\right)\\
&\qquad\qquad +i\sqrt{2}|M|^{-1}\left(\tilde{\mathcal{J}}_{[I}{}^K\partial_kM_{K|J]} - \frac{i}{4}\frac{\partial_k\bar{\tau}}{\Im\mathrm{m}\tau}\tilde{M}_{IJ}\right)\gamma^k\gamma_i\gamma^0\epsilon^J = 0
\end{aligned} \tag{3.110}$$

$$\gamma^k\left[\frac{d\tau}{\Im\mathrm{m}\tau}\left(\epsilon_I + i\sqrt{2}\gamma^0\frac{M_{IJ}}{|M|}\epsilon^J\right) + 2\sqrt{2}|M|^{-1}D\tilde{M}_{IJ}\gamma^0\epsilon^J\right]_k = 0 \tag{3.111}$$

We impose the following constraint on the $SU(4)$ spinors

$$\epsilon_I + i\sqrt{2}\gamma^0 \frac{M_{IJ}}{|M|}\epsilon^J = 0 \, . \tag{3.112}$$

We call this constraint "the $SU(4)$ reality condition". It is a kind of reality condition because it relates the $SU(4)$ spinors with their complex conjugates. This condition implies, together with (3.97),

$$M_{[IJ}\epsilon_{K]} = 0 \, , \quad \mathcal{J}_I{}^J \epsilon_J = \epsilon_I \, . \tag{3.113}$$

For generic (i.e. not built from already-known Killing spinors) scalars M_{IJ} the above relation would be a constraint breaking 1/2 of the supersymmetries to be imposed on the Killing spinors whenever $M^{KL}\mathcal{D}_i M_{KL} \neq 0$. The counting of unbroken supersymmetries is, however, a bit more subtle and depends on the triviality of the $U(1)$ connection ξ: if ξ is a total derivative the projection $\mathcal{J}^I{}_J$ is constant and a global $SU(4)$ rotation suffices to set to zero two of the chiral Killing spinors. This is the procedure followed by Tod in Ref. [5], where he solved the constant $\mathcal{J}^I{}_J$ (*internally rigid*) case by setting to zero two of the spinors, breaking the explicit $SU(4)$ covariance of the solutions. The solutions found by Tod preserve, then, generically, 1/4 of the supersymmetries[6]. If $\mathcal{J}^I{}_J$ is not constant, ξ is non-trivial and the 4 Killing spinors cannot be related by global $SU(4)$ rotations, but we are now going to see that this case can also be solved introducing a new projection on the Killing spinors which also reduces the amount of generically preserved supersymmetries to 1/4.

After imposing the condition (3.112) the KSEs becomes

$$\partial_0 \epsilon_I = 0 \tag{3.114}$$

$$\nabla_i \epsilon_I - \frac{1}{2|M|^2} M^{KL}\partial_i M_{KL} - \partial_i \mathcal{J}_I{}^J \epsilon_J = 0 \tag{3.115}$$

$$\not{\partial} \mathcal{J}_I{}^J \epsilon_J = 0 \, . \tag{3.116}$$

The solution to (3.114) are static spinors. Eq. (3.115) can be written in such a way that the static Killing spinors satisfy the equations

$$\mathcal{J}_I{}^J \left(\nabla_i - i\xi_i\right) \epsilon_J = 0 \tag{3.117}$$

$$\not{\partial} \mathcal{J}_I{}^J \epsilon_J = 0 \, , \tag{3.118}$$

where we have rescaled the $SU(4)$ spinors by a factor $\sqrt{|M|}$.

The $SU(4)$ reality condition (3.112) can be derived from the following two constraints

$$\epsilon_A + e^{i\lambda}\gamma^0 \epsilon_{AB}\epsilon^B = 0 \, , \tag{3.119}$$

$$\mathcal{J}_I{}^J \phi^A_J = \phi^A_I \, . \tag{3.120}$$

[6]The conditions under which 1/2 of the supersymmetries are preserved were studied in Ref. [13].

Note that the presence of λ in the equation (3.119) ensures the $SL(2,\mathbb{R})$ and $U(1) \subset U(2)$ convariance. In order to symplify the notation we absorb the λ factors by a change on the spinors

$$\epsilon_A{}' = e^{-\frac{i}{2}\lambda}\epsilon_A \tag{3.121}$$

(this is not a symmetry transformation, just a change of variables). However, we must be careful with the fact that the new spinors $\epsilon_A{}'$ are inert under $SL(2,\mathbb{R})$ and $U(1) \subset U(2)$. Therefore, the constraint on scalars and spinors are, supriming the primes from now on,

$$\epsilon_A + \gamma^0 \epsilon_{AB} \epsilon^B = 0 \ , \tag{3.122}$$

$$\mathcal{J}_I{}^J \phi^A_J = \phi^A_I \ . \tag{3.123}$$

Constraint (3.122), which we call "the U(2) reality condition", says that one of the two components of the $U(2)$ spinors is proportional to the complex conjugate of the other one,

$$\epsilon_2 = \gamma^0 \epsilon^1 \tag{3.124}$$

and this a sort of "Majorana condition" in the $U(2)$ space.

In the $U(2)$ formalism we must assume that the two $SU(4)$ vectors ϕ^A_I are linearly independent, otherwise we would be in the null case. According to (3.123), the two $SU(4)$ vectors ϕ^A_I are the two eigenvectors of $\mathcal{J}_I{}^J$ with eigenvalues $+1$, the other two eigenvectors of $\mathcal{J}_I{}^J$ correspond to the eigenvalue zero. Thus the theorem of the spectral decomposition ensures that the action of $\mathcal{J}_I{}^J$ on any $SU(4)$ vector v_I is given by

$$\mathcal{J}_I{}^J v_J = \phi^A_I \phi^J_A v_J \ . \tag{3.125}$$

Therefore we may impose the condition

$$\phi^A_I \phi^J_B = \mathcal{J}_I{}^J \tag{3.126}$$

as a general constraint equivalent to (3.123). It is also a solution of (B.62).

Now we study the KSEs (3.117) and (3.118), starting with (3.117). It is equivalent to the equations

$$\nabla_i \epsilon_A - \mathcal{A}_{i\,A}{}^B \epsilon_B - \frac{i}{2}\left(2\xi - i\zeta - d\lambda\right)_i \epsilon_A = 0 \tag{3.127}$$

$$\left(\mathcal{J}_I{}^J - \phi^A_I \phi^J_A\right) \partial_i \phi^B_J \epsilon_B = 0 \ . \tag{3.128}$$

The last equation is automatically solved due to (3.123).

The equation (3.127) can be decomposed into two parts using the reality condition (3.122) on the spinors. Taking the complex conjugate of (3.127) and using (3.122) we obtain the equation

$$\nabla_i \epsilon_A - \mathcal{A}_{i\,A}{}^B \epsilon_B + \frac{i}{2}\left(2\xi - i\zeta - d\lambda\right)_i \epsilon_A = 0 \ . \tag{3.129}$$

The two equations (3.127) and (3.129) are equivalent to

$$\nabla_i \epsilon_A - \mathcal{A}_{i\,A}{}^B \epsilon_B = 0 \tag{3.130}$$

$$(2\xi - i\zeta - d\lambda)\,\epsilon_A = 0\ . \tag{3.131}$$

The obvious solution to the equation (3.131) is that the $SL(2,\mathbb{R})$ connection ξ and the $U(1) \subset U(2)$ connection ζ are in the same cohomology class,

$$2\xi - i\zeta = d\lambda \tag{3.132}$$

which should be an identity.

The total system of KSEs, including the dilatino equation, is

$$\nabla_i \epsilon_A - \mathcal{A}_{i\,A}{}^B \epsilon_B = 0 \tag{3.133}$$

$$\not{\partial}\mathcal{J}_I{}^J \phi_J^A \epsilon_A = 0 \tag{3.134}$$

subject to

$$\epsilon_A + \gamma^0 \epsilon_{AB} \epsilon^B = 0\ , \tag{3.135}$$

$$\phi_I^A \phi_A^J = \mathcal{J}_I{}^J \tag{3.136}$$

$$\phi_I^A \phi_B^I = \delta_B^A\ . \tag{3.137}$$

The integrability condition of the equation (3.133) is

$$\frac{1}{4} R_{kl} \gamma^{kl} \epsilon_A - \mathcal{R}_A{}^B \epsilon_B = 0 \tag{3.138}$$

where

$$R_k{}^l = do_k{}^l + o_k{}^m \wedge o_m{}^l\ , \tag{3.139}$$

and says that the effect of the spin connection on the spinors is equivalent to the action of the $SU(2)$ connection.

The constraints (3.135) and (3.136) are constraints for the Killing spinors ϵ_I. Hence we should count the number of supersymmetries which are broken by them. Constraint (3.135) fix one of the $U(2)$ spinors in terms of the other one. On the other hand, the projector $\mathcal{J}_I{}^J$ is an hermitean matrix of rank 2, such that one half of the ϕ_I^A are supressed by constraint (3.136), which is equivalent to (3.123). Since the Killing spinors ϵ_I are the product of ϕ_I^A and ϵ_A, one quarter of the supersymmetries are preserved by constraints (3.135) and (3.136). However, there could be further supersymmetries broken by the eqs. (3.133) and (3.134).

We analyze the KSE (3.133), starting with its integrability condition (3.138). We introduce the curvatures of the $SU(2)$ connection in the adjoint representation

$$\mathcal{R}^x = d\mathcal{A}^x - \frac{1}{2}\epsilon^{xyz}\mathcal{A}_y \wedge \mathcal{A}_z\ , \tag{3.140}$$

such that

$$\mathcal{R}^x = i\sigma^{x\,B}_A \mathcal{R}_B{}^A \tag{3.141}$$

$$\mathcal{R}_A{}^B = -\frac{i}{2}\sigma^{x\,B}_A \mathcal{R}_x \ . \tag{3.142}$$

Due to the correspondence (3.89) between the spin and $SU(2)$ connections, the curvatures are related similarly

$$R^{xy} = \epsilon^{xyz}\mathcal{R}_z \ . \tag{3.143}$$

The integrability condition (3.138) becomes

$$\mathcal{R}_x \left(\gamma^{0x}\delta_A{}^B - \sigma^{x\,B}_A\right)\epsilon_B = 0 \ . \tag{3.144}$$

We may state the general solution to this condition by means of the projectors

$$\Pi^{x\,B}_A \equiv \frac{1}{2}\left(\delta_A{}^B - \gamma^{0(x)}\sigma^{(x)\,B}_A\right) \ , \tag{3.145}$$

where the notation (x) means that there is not sum over x. Then the solution to the integrability condition is

$$\Pi^{x\,B}_A \epsilon_B = 0 \quad \text{if} \quad \mathcal{R}_x \neq 0 \ . \tag{3.146}$$

Now we move to the KSE (3.133). After using the correspondence (3.89) to replace the spin connection, the KSE becomes

$$\partial_i \epsilon_A - \frac{i}{2}\mathcal{A}_x \left(\gamma^{0x}\delta_A{}^B - \sigma^{x\,B}_A\right)\epsilon_B = 0 \tag{3.147}$$

and it is solved by constant spinors that satisfy

$$\Pi^{x\,B}_A \epsilon_B = 0 \quad \text{if} \quad \mathcal{A}_x \neq 0 \ . \tag{3.148}$$

Coming back to the original Killing spinors, they are given by

$$\epsilon_I = \sqrt{|M|}e^{\frac{i}{2}\lambda}\phi^A_I \epsilon^{(0)}_A \ , \tag{3.149}$$

where

$$\phi^A_I \phi^J_A = \mathcal{J}_I{}^J \tag{3.150}$$

$$\phi^A_I \phi^I_B = \delta^A_B \tag{3.151}$$

$$\epsilon^{(0)}_A + \gamma^0 \epsilon_{AB}\epsilon^{(0)B} = 0 \tag{3.152}$$

$$\Pi^{x\,B}_A \epsilon^{(0)}_B = 0 \quad \text{if} \quad \mathcal{A}_x \neq 0 \tag{3.153}$$

and e^λ is an arbitraty phase charged under $SL(2,\mathbb{R})$ and $U(2)$ with weight $+1$

The dilatino KSE

$$\not{\partial}\mathcal{J}_I{}^J \phi^A_J \epsilon_A = 0 \tag{3.154}$$

is still there, but seemly it is solved with the above conditions. For the case of Tod's configurations (the flat ones) the projector $\mathcal{J}$ is constant. For the holomorphic case (the $U(1)$ holonomy) the projection (3.153) implies the dilatino equation, as we will show in the next section.

Supersymmetric solutions

Finally, it is convenient to rewrite the equations of motion of the vector and scalar fields in differential-form language[7]:

$$\hat{\vec{\mathcal{E}}}^{IJ} \equiv \vec{\mathcal{E}}^{IJ}{}_{\mu} dx^{\mu} = -\star d\vec{F}^{IJ} = \begin{pmatrix} \hat{\mathcal{E}}^{IJ} \\ \hat{\mathcal{B}}^{IJ} \end{pmatrix}, \tag{3.155}$$

$$\hat{\mathcal{E}} \equiv \mathcal{E}\hat{V}, \tag{3.156}$$

where $\vec{\mathcal{E}}^{IJ}{}_{\mu}$ is the $SL(2,\mathbb{R})$ doublet formed by the Maxwell and Bianchi identities:

$$\vec{\mathcal{E}}^{IJ\,\mu} \equiv \begin{pmatrix} \mathcal{E}^{IJ\,\mu} \\ \mathcal{B}^{IJ\,\mu} \end{pmatrix} = \begin{pmatrix} \nabla_{\nu}\star\tilde{F}^{IJ\,\nu\mu} \\ \nabla_{\nu}\star F^{IJ\,\nu\mu} \end{pmatrix}. \tag{3.157}$$

we find the following two equations for M_{IJ} and τ:

$$\star\hat{\vec{\mathcal{E}}}^{IJ} = \tfrac{1}{2} d\star\left[\frac{\mathcal{M}Sd\vec{E}_{IJ}}{|M|^2}\wedge\hat{V}\right] + d\vec{E}_{IJ}\wedge\Omega, \tag{3.158}$$

$$\frac{\star\hat{\mathcal{E}}^*}{|M|^2} = -\mathcal{D}\star\left[\frac{d\tau}{|M|^2\Im\mathrm{m}\,\tau}\wedge\hat{V}\right] + 2i\frac{d\tau}{\Im\mathrm{m}\,\tau}\wedge\Omega + 2i\frac{\tilde{M}_{IJ}}{|M|^2}d\star\left(\frac{dM^{IJ}}{|M|^2}\wedge\hat{V}\right) \tag{3.159}$$

These equations can be now be combined (this is the reason behind the introduction of V into the equation for τ and the use of differential forms) and simplified. Using the new variables N_{IJ} defined by

$$N_{IJ} = \sqrt{\Im\mathrm{m}\tau}M_{IJ}, \qquad |N|^2 = N^{IJ}N_{IJ} = \Im\mathrm{m}\tau|M|^2, \tag{3.160}$$

we construct a new combination of equations that we call $\hat{a}^{IJ}$

$$\hat{a}^{IJ} \equiv \frac{1}{2\sqrt{2}\Im\mathrm{m}\tau}(\tau\hat{\mathcal{B}}^{IJ} - \hat{\mathcal{E}}^{IJ}) - \tfrac{i}{2}\frac{(N^{IJ}+\tilde{N}^{IJ})}{|N|^2}\hat{\mathcal{E}}^*, \tag{3.161}$$

and, which, after some massaging, is going to have a much simpler form. To present in compact form the equations of motion we define these two equations

$$n^{IJ} \equiv (\nabla_{\mu} + 4i\xi_{\mu})\left(\frac{\partial^{\mu}N^{IJ}}{|N|^2}\right), \tag{3.162}$$

$$e^* \equiv (\nabla_{\mu} + 4i\xi_{\mu})\left(\frac{\partial^{\mu}\tau}{|N|^2}\right), \tag{3.163}$$

[7]We add hats to denote differential forms.

and, in terms of them, we have, switching again from differential form notation to tensor notation,

$$a^{IJ} = n^{IJ} - \frac{N^{IJ} + \tilde{N}^{IJ}}{|N|^2}\tilde{N}_{KL}n^{KL}\,, \tag{3.164}$$

$$\mathcal{B}^{IJ\,a} = \sqrt{2}V^a\left\{\frac{N^{IJ} + \tilde{N}^{IJ}}{|N|^2}\Re\mathrm{e}\,\mathcal{E} - i(a^{IJ} - \tilde{a}^{IJ})\right\}\,, \tag{3.165}$$

$$\mathcal{E}^{IJ\,a} = \sqrt{2}V^a\left\{\frac{N^{IJ} + \tilde{N}^{IJ}}{|N|^2}\Re\mathrm{e}\,(\tau\mathcal{E}) - i(\bar{\tau}a^{IJ} - \tau\tilde{a}^{IJ})\right\}\,. \tag{3.166}$$

$$\mathcal{E} = |M|^2 e + 2i\tilde{N}^{KL}n_{KL}\,. \tag{3.167}$$

The combination $|N|^{-2}d\tau$ has $U(1)$ charge -4 and, thus, the second equation is just a $U(1)$-covariant divergence, the covariant derivative being constructed with the ξ connection. The first equation has a similar form and, although $\frac{dN_{IJ}}{|N|^2}$ does not transform covariantly under $SL(2,\mathbb{R})$, the equation is $SL(2,\mathbb{R})$-covariant up to terms proportional to the second equation.

These are equations for the scalars M_{IJ} and τ and involve implicitly the spacetime metric, which is the only field not determined by them. We need to study now the Einstein equations and, to do it, it is convenient to choose coordinates adapted to the timelike Killing vector V.

Since $|M|$ is in principle determined by the above equations, we only need to find equations for ω and γ.

Using the conformastationary metric we can reduce all the equations to equations in the 3 spatial dimensions with the metric γ. To start with, the equations n^{IJ} and e defined in Eqs. (3.162) and (3.163) can be expressed in terms of

$$n^{IJ}_{(3)} \equiv (\nabla_{\underline{i}} + 4i\xi_{\underline{i}})\left(\frac{\partial^{\underline{i}}N^{IJ}}{|N|^2}\right)\,, \tag{3.168}$$

$$e^{*}_{(3)} \equiv (\nabla_{\underline{i}} + 4i\xi_{\underline{i}})\left(\frac{\partial^{\underline{i}}\tau}{|N|^2}\right)\,, \tag{3.169}$$

where all the objects are now 3-dimensional with metric γ, by

$$n^{IJ} = -|M|^2 n^{IJ}_{(3)}\,, \qquad e = -|M|^2 e_{(3)}\,. \tag{3.170}$$

The equation (3.84) for the 1-form ω that enters the conformastationary metric reduces to

$$f_{ij} = 4|M|^{-2}\epsilon_{ijk}(\xi_k - Q_k)\,, \qquad f_{\underline{ij}} \equiv 2\partial_{[\underline{i}}\omega_{\underline{j}]}\,. \tag{3.171}$$

Then, we can express all the equations of motion in terms of these two equations plus the equation[8]

$$e_{ij} \equiv R_{ij}(\gamma) - 2\partial_{(i}\left(\frac{N^{IJ}}{|N|}\right)\partial_{j)}\left(\frac{N_{KL}}{|N|}\right)(\delta^{KL}{}_{IJ} - \mathcal{J}^K{}_I\mathcal{J}^L{}_J)\,, \tag{3.172}$$

as follows:

$$\mathcal{E}_{00} = |M|^2\left[|M|^2\Im\mathrm{m}e^*_{(3)} - 2\Re\mathrm{e}\,(N_{KL}n^{KL}_{(3)}) + \tfrac{1}{2}e_k{}^k\right]\,, \tag{3.173}$$

$$\mathcal{E}_{0i} = 0\,, \tag{3.174}$$

$$\mathcal{E}_{ij} = |M|^2(e_{ij} - \tfrac{1}{2}\delta_{ij}e_k{}^k)\,, \tag{3.175}$$

$$\mathcal{B}^{IJ\,a} = -\sqrt{2}|M|^2V^a\left\{\frac{N^{IJ}+\tilde{N}^{IJ}}{\Im\mathrm{m}\tau}\Re\mathrm{e}\,e_{(3)} - i(n^{IJ}_{(3)} - \tilde{n}^{IJ}_{(3)})\right\}\,, \tag{3.176}$$

$$\mathcal{E}^{IJ\,a} = -\sqrt{2}|M|^2V^a\left\{\frac{N^{IJ}+\tilde{N}^{IJ}}{\Im\mathrm{m}\tau}\Re\mathrm{e}\,(\tau e_{(3)}) - i(\bar{\tau}n^{IJ}_{(3)} - \tau\tilde{n}^{IJ}_{(3)})\right\}\,. \tag{3.177}$$

$$\mathcal{E} = -|M|^2\left[|M|^2e_{(3)} + 2iN_{KL}\tilde{n}^{KL}_{(3)}\right]\,. \tag{3.178}$$

We are now ready to check whether these equations satisfy the relations expressed in Eqs. (3.45-3.47). It is immediate to see that they do if the following conditions are satisfied off-shell:

$$e_{ij} = 0\,, \tag{3.179}$$

$$|M|^2\Re\mathrm{e}\,(e_{(3)}) - 2\Im\mathrm{m}(N_{IJ}n^{IJ}_{(3)}) = 0\,. \tag{3.180}$$

The first equation determines the 3-dimensional matric γ as a function of the scalars N^{IJ} and says that γ is Ricci-flat is the projection $\mathcal{J}^I{}_J$ is constant. The second equation can be rewritten in the form

$$\nabla_{\underline{i}}\left(\frac{Q^{\underline{i}} - \xi^{\underline{i}}}{|M|^2}\right) = 0\,, \tag{3.181}$$

[8]This equation should be compared with Eq. (3.81) in which the antisymmetric part of the same combination appears.

and is the integrability condition of Eq. (3.171) for the 1-form ω, whose existence we have assumed throughout all this analysis. Thus, it is not so much a necessary condition for supersymmetry as it is a necessary condition for the whole problem to be well defined.

Let us summarize the results of this section: we have seen that, in the timelike case at hands, field configurations with a metric of the form Eq. (3.73), vector field strengths of the form Eq. (3.65) and any complex scalar τ, and satisfying Eqs. (3.179) and (3.180) satisfy all the integrability conditions of the Killing spinor equations.

On the other hand, all the equations of motion, including the Bianchi identities, are satisfied if the equations

$$e^*_{(3)} = 0\,, \qquad n^{IJ}_{(3)} = 0\,, \qquad e_{ij} = 0\,, \tag{3.182}$$

(were $e^*_{(3)}$ and $n^{IJ}_{(3)}$ are defined in Eq. (3.169) and Eq. (3.168)) are satisfied, and automatically the integrability conditions are also satisfied.

3.4.3 Some explicit examples

The holomorphic case

Now we study a particular class of configurations, which we call "the holomorphic" case. They are given by two scalars M_{IJ},

$$M_{12}(x,z,\bar{z}) = e^{i\lambda}k_1(z)\ , \tag{3.183}$$

$$M_{13}(x,z,\bar{z}) = e^{i\lambda}k_2(z) \tag{3.184}$$

where we are using the system of local spatial coordinates $(x,z,\bar{z})$. The other M_{IJ} are zero. Since we have only two of the M_{IJ} non-zero, the constraint (3.97) on them is automatically solved. Therefore k_1 and k_2 are arbitrary holomorphic functions.

The $SL(2,\mathbb{R})$ connection is

$$\xi = -\frac{i}{4}\left(\partial_z U dz - \partial_{\bar{z}} U d\bar{z}\right) + \frac{1}{2}d\lambda \tag{3.185}$$

and its curvature is

$$d\xi = \frac{i}{2}\partial^2_{z\bar{z}} U dz \wedge d\bar{z} \tag{3.186}$$

where

$$U = \ln|k|^2\ , \quad |k|^2 \equiv 2\left(|k_1|^2 + |k_2|^2\right) \tag{3.187}$$

The projector $\mathcal{J}$ is

$$[\mathcal{J}_I{}^J] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2|k|^{-2}|k_1|^2 & 2|k|^{-2}k_1k^2 & 0 \\ 0 & 2|k|^{-2}k^1k_2 & 2|k|^{-2}|k_2|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.188}$$

A configuration of ϕ^A_I which solves the constraints (3.136) and (3.137) is

$$[\phi^A{}_I] = \begin{pmatrix} \frac{i}{\sqrt{2}} & |k|^{-1}k_1 & |k|^{-1}k_2 & 0 \\ -\frac{i}{\sqrt{2}} & |k|^{-1}k_1 & |k|^{-1}k_2 & 0 \end{pmatrix} . \tag{3.189}$$

For this configuration, all of the $U(2)$ components of the connection $\phi^I_A d\phi^A_I$ are the same, then

$$\zeta = \frac{1}{2}\left(\partial_z U dz - \partial_{\bar{z}} U d\bar{z}\right) , \tag{3.190}$$

$$[\mathcal{A}_A{}^B] = \frac{1}{2}\begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix} \tag{3.191}$$

The $SU(2)$ connection has only one independent component, hence it works as a $U(1)$ connection. Consequently, the curvatures are

$$d\zeta = -\partial^2_{z\bar{z}} U dz \wedge d\bar{z} , \tag{3.192}$$

$$[\mathcal{R}_A{}^B] = \frac{1}{2}\begin{pmatrix} 0 & d\zeta \\ d\zeta & 0 \end{pmatrix} . \tag{3.193}$$

The only non-zero component of the $SU(2)$ connection in the adjoint representation is that of the σ^1 matrix,

$$\mathcal{A}^{x=1} = i\zeta , \tag{3.194}$$

$$\mathcal{R}^{x=1} = id\zeta . \tag{3.195}$$

According to the correspondence (3.89) between the spin and $SU(2)$ connections, the spin connection becomes a $U(1)$ connection

$$o^{23} = -i\zeta \tag{3.196}$$

An the spatial metric has $U(1)$ holonomy. The three dimensional euclidean metrics with $U(1)$ holonomy factorize as the product of one- and two-dimensional metrics

$$h_{ij}dx^i dx^j = dx^2 + 2e^U dz d\bar{z} , \tag{3.197}$$

where U is given by (3.187).

The Killing spinors are given by

$$\epsilon_I = \sqrt{|k|}e^{\frac{i}{2}\lambda}\phi_I^A\epsilon_A^{(0)} \tag{3.198}$$

where λ is arbitrary, ϕ_I^A is given by (3.189) and the constant spinors are subject to

$$\epsilon_A^{(0)} + \gamma^0\epsilon_{AB}\epsilon^{(0)B} = 0 \,, \tag{3.199}$$

$$(d\zeta)\,\Pi^1_A{}^B\epsilon_B^{(0)} = (d\zeta)\,\frac{1}{2}\left(\delta_A{}^B - \gamma^{01}\sigma^1_A{}^B\right)\epsilon_B^{(0)} = 0 \,. \tag{3.200}$$

These two conditions are solved by

$$\epsilon_2 = \gamma^0\epsilon^1 \tag{3.201}$$

$$\epsilon_2 = \gamma^{01}\epsilon_1 \,, \quad (\text{if } \; d\zeta \neq 0) \tag{3.202}$$

and in turn these conditions are solved by a single spinor which satisfy a sort of reality condition,

$$\epsilon_1 = \epsilon \,, \quad \epsilon_2 = \gamma^0\epsilon^* \,, \tag{3.203}$$

$$\epsilon^* = \gamma^1\epsilon \,, \quad (\text{if } \; d\zeta \neq 0) \,. \tag{3.204}$$

The sector of $SU(2)$ spinors has only one quarter of the degree of freedom and the matrix $[\phi^A{}_I]$ has rank 2, therefore this class of configurations preserve 1/8 of the supersymmetries. If $d\zeta = 0$, the spinor ϵ does not necessarily satisfy the reality condition and hence in that case 1/4 of the supersymmetries are preserved (the Tod case).

Finally we check the dilatino equation. In general this equation can be written as

$$\left(\partial_i\phi_I^A - \frac{1}{2}\zeta_i\phi_I^A + \frac{i}{2}\mathcal{A}_{iy}\sigma^y_B{}^A\phi_I^B\right)\gamma^i\epsilon_A = 0 \,. \tag{3.205}$$

In the holomorphic case this equation takes the form

$$\left[\partial_i\phi_I^A - \frac{1}{2}\zeta_i\left(\delta_B{}^A + \sigma^1_B{}^A\right)\phi_I^B\right]\gamma^i\epsilon_A = 0 \,. \tag{3.206}$$

Now let us see in detail each one of the $SU(4)$ components of this equation. For $I=4$ this is trivially zero. For $I=1$, $\phi^A_{I=1}$ is constant and it is the null eigenvector (zero eigenvalue) of $1+\sigma^1$, hence the equation is solved. For $I=2,3$ the $\phi's$ are

$$\left[\phi_I^A\right] = |k|^{-1}k_I\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.207}$$

and this $U(2)$ vector is the eigenvector of $1+\sigma^1$ with eigenvalue $+2$. Then the equation for $I=2,3$ becomes

$$\left[\partial_i\left(|k|^{-1}k_I\right) - \zeta_i|k|^{-1}k_I\right]\gamma^i\left(\epsilon_1+\epsilon_2\right) \,. \tag{3.208}$$

In principle, the expression between square brackets has only two (curved) components $i=z,\bar{z}$. However, it is easy to see that the $\bar{z}$ component vanishes. Since in this case the metric has $U(1)$ holonomy the vielbeins matrix is diagonal. Then the z component of the above equation is proportional to

$$\gamma^{z}\left(\epsilon_1+\epsilon_2\right) \tag{3.209}$$

which, after the projection (3.202), is

$$\gamma^{z}\left(1+\gamma^{01}\right)\epsilon_1 \tag{3.210}$$

and this expression is automatically zero due to the chirality of the spinors.

According to the recipe of the previous section, our first step in finding supersymmetric configurations and solutions is to find the complex scalars M^{IJ} satisfying $\varepsilon^{IJKL}M_{IJ}M_{KL}=0$ and such that ξ can be written in the form Eq. (3.132). The first condition can be easily met, for instance, by taking only M_{12}, M_{13} and M_{23} non-vanishing, but we prefer not to make any specific choice that would break $SU(4($ covariance. The second condition can be solved by the following *Ansatz*

$$M_{IJ}=e^{i\lambda(x,z,\bar{z})}M(x,z,\bar{z})k_{IJ}(z)\,,\qquad M=M^{*}\,,\qquad \lambda=\lambda^{*}\,,\qquad \varepsilon^{IJKL}k_{IJ}k_{KL}=0\,, \tag{3.211}$$

which give a connection ξ of the form Eq. (3.132) with

$$U=+\ln|k|\,,\qquad |k|^2\equiv k^{IJ}(\bar{z})k_{IJ}(z)\,, \tag{3.212}$$

and satisfies automatically the integrability condition Eq. (3.179).

Solving the integrability condition Eq. (3.181) is considerably more difficult and considering solutions (instead of general configurations) simplifies the problem. We have found three families of solutions.

1. If the k_{IJ} are constants, then, normalizing $|k|^2=1$ for simplicity, $\xi=\frac{1}{2}d\lambda$ and $U=0$. This is the case considered by Tod in Ref. [5] and studied in detail in Ref. [13]. Tod took advantage of the fact that $d\xi=0$ implies that $\mathcal{J}^{I}{}_{J}$ is constant and a global $SU(4)$ rotation can be used to set to zero two of the ϵ_Is. We will not do so, as this breaks the explicit $SU(4)$ covariance, but our results are, of course, equivalent.

   Eq. (3.168) takes the form

$$\partial_{\underline{i}}\partial_{\underline{i}}\mathcal{H}_1=0\,,\qquad \mathcal{H}_1\equiv[\sqrt{\Im\mathrm{m}\,\tau}e^{-i\lambda}M]^{-1}\,, \tag{3.213}$$

and is solved by any arbitrary complex harmonic function $\mathcal{H}_1$.

Using the above equation, Eq. (3.169) takes the form

$$\partial_{\underline{i}}\partial_{\underline{i}}(\mathcal{H}_1\tau) = 0\,, \tag{3.214}$$

which is solved by

$$\tau = \mathcal{H}_1/\mathcal{H}_2\,, \qquad \partial_{\underline{i}}\mathcal{H}_2 = 0\,, \tag{3.215}$$

another arbitrary complex harmonic function. The pair of harmonic functions and the constants determine completely the solutions. In particular

$$|M|^{-2} = M^{-2} = \Im\mathrm{m}(\bar{\mathcal{H}}_2\mathcal{H}_1)\,. \tag{3.216}$$

2. If $e^{i\lambda} = M = 1$, the integrability condition Eq. (3.181) can be solved by taking τ constant. The only non-trivial equation of motion, Eq. (3.168) is solved using the holomorphicity of the k_{IJ}s. The metric takes the form

$$ds^2 = |k|^2(dt + \omega_x dx) - |k|^{-2}dx^2 - 2dzd\bar{z}\,, \tag{3.217}$$

where ω_x satisfies

$$\partial_z\omega_x - \partial_x\omega_z = \partial_{\bar{z}}|k|^{-2}\,, \quad \partial_{\bar{z}}\omega_x - \partial_x\omega_{\bar{z}} = \partial_z|k|^{-2}\,, \quad \partial_{\bar{z}}\omega_z - \partial_z\omega_{\bar{z}} = 0\,. \tag{3.218}$$

The metric and the supersymmetry projectors indicate that these solutions describe stationary strings lying along the coordinate x, in spite of the trivial axion field, which is the dual of the Kalb-Ramond 2-form B that couples to strings. Observe, however, that the duality relation is not simply $dB = \star da$: there are terms quadratic in the field strengths involved in the duality which must render B non-trivial.

The metric and the vector fields involved depends strongly on the choice of holomorphic k_{IJ}s. It is instructive to have an example completely worked out.

Let us consider the simplest case: only $k_{12} = \frac{1}{\sqrt{2}z}$ non-trivial. This allows us to set $\omega_z = \omega_{\bar{z}} = 0$. Then, $|k|^2 = |z|^{-2}$ and $\omega_x = 2\Re\mathrm{e}(z^2)$ and the full solution is given by

$$
\begin{aligned}
ds^2 &= \frac{1}{|z|^2}[dt + 2\Re\mathrm{e}(z^2)dx]^2 - |z|^2 dx^2 - 2dzd\bar{z}\,, \\
F_{12} &= -\frac{\sqrt{2}e^{\phi_0/2}}{z^2}\{[dt + 2\Re\mathrm{e}(z^2)dx] \wedge dz - i\star[[dt + 2\Re\mathrm{e}(z^2)dx] \wedge dz]\} = (F_{34})^*\,, \\
\tau &= \tau_0\,.
\end{aligned} \tag{3.219}
$$

3. The only solutions that we have found with λ and the $k_{IJ}(z)$s simultaneously nontrivial have just $\lambda = \lambda(x)$ and $M = M(x)$ and are a superposition of the solutions with constant k_{IJ} and the solutions with constant λ in which these functions depend only on mutually transversal directions.

   Thus, these solutions depend on holomorphic functions $k_{IJ}(z)$ chosen with the same criteria as in the previous case, and a pair of complex functions $\mathcal{H}_1, \mathcal{H}_2$ linear in x such that $\Im\mathrm{m}\,\tau > 0$, and the metric is given by

$$
ds^2 = (M|k|)^2(dt + \omega_x dx) - (M|k|)^{-2}dx^2 - 2M^{-2}dzd\bar{z}\,, \tag{3.220}
$$

   where M is again given by Eq. (3.216).

3.4.4 The null case

As we have mentioned before, the null case was completely solved by Tod in Ref. [5], but we include it her for the sake of completeness.

As explained in Appendix B.1.2, in the null case all the spinors are proportional $\epsilon_I = \phi_I\epsilon$. In the $N = 4, d = 4$ case at hands, ϵ_I has a $U(1)$ charge under $SL(2,\mathbb{R})$ transformations that has to be distributed between ϕ_I and ϵ. We choose to have the ϕ_I uncharged. Had we chosen to have ϕ_I is charged with charge $q_\phi \neq 0$, then the real 1-form

$$
\zeta \equiv i\phi_I d\phi^I\,, \tag{3.221}
$$

would transform as a $U(1)$ connection under $SL(2,\mathbb{R})$ transformations as well and would play a role analogous to that of the connection ξ in the timelike case. With our choice, ζ is just a $U(1)$ connection under the transformations Eq. (B.41) and covariantizes with respect to them the expressions that involve ϵ.

We are now going to substitute $\epsilon_I = \phi_I\epsilon$ into the KSEs and we are going to use the normalization condition to split the KSEs into three algebraic and one differential equation for ϵ. One of the algebraic equations for ϵ will be a differential equation for ϕ_I.

The substitution yields immediately

$$
\begin{aligned}
\mathcal{D}_\mu \phi_I \epsilon + \phi_I \mathcal{D}_\mu \epsilon - \tfrac{i}{2\sqrt{2}} \sqrt{\Im\mathrm{m}\,\tau} F_{IJ}{}^+{}_{\mu\nu} \phi^J \gamma^\nu \epsilon^* &= 0\,, && (3.222)\\
\phi_I \frac{\not{\partial}\tau}{\Im\mathrm{m}\,\tau} \epsilon - \tfrac{1}{2\sqrt{2}} \sqrt{\Im\mathrm{m}\,\tau}\, \not{F}_{IJ}{}^- \phi^J \epsilon^* &= 0\,. && (3.223)
\end{aligned}
$$

Acting on Eq. (3.222) with ϕ^I leads to

$$
\mathcal{D}_\mu \epsilon = -\phi^I \mathcal{D}_\mu \phi_I \epsilon\,, \tag{3.224}
$$

which takes the form

$$
\tilde{\mathcal{D}}_\mu \epsilon \equiv (\mathcal{D}_\mu + i\zeta_\mu)\epsilon = 0\,, \tag{3.225}
$$

and becomes the only differential equation for ϵ. We have defined the derivative $\tilde{\mathcal{D}}$ covariant with respect to $SL(2,\mathbb{R})$ and $U(1)$ local rotations under which ϵ and ϕ_I have charges $+1$ and -1, respectively. Using Eq. (3.225) into Eq. (3.222) to eliminate $\mathcal{D}_\mu \epsilon$ we obtain

$$
\tilde{\mathcal{D}} \phi_I \epsilon - \tfrac{i}{2\sqrt{2}} \sqrt{\Im\mathrm{m}\,\tau} F_{IJ}{}^+{}_{\mu\nu} \phi^J \gamma^\nu \epsilon^* = 0\,, \tag{3.226}
$$

which is one of the algebraic constraints for ϵ and is a differential equation for ϕ_I.

Acting with ϕ^I on Eq. (3.223) we see that it splits into two algebraic constraints for ϵ:

$$
\begin{aligned}
\not{\partial}\tau \epsilon &= 0\,, && (3.227)\\
\not{F}_{IJ}{}^- \phi^J \epsilon^* &= 0\,. && (3.228)
\end{aligned}
$$

Finally, we add to the system an auxiliary spinor η, introduced in Appendix B.1.2, with charges opposite to those of ϵ. The normalization condition Eq. (B.40) will be preserved if and only if η satisfies a differential equation of the form

$$
\tilde{D}_\mu \eta + a_\mu \epsilon = 0\,, \tag{3.229}
$$

where a_μ is, in principle, an arbitrary vector with the right charges that transforms under the redefinitions Eqs. (B.49) and (B.50) as a connection

$$a'_\mu = a_\mu + \partial_\mu \delta\,. \tag{3.230}$$

In practice, however, a_μ cannot be completely arbitrary since the integrability conditions of the differential equation of η have to be compatible with those of the differential equation for ϵ and this requirement will determine a_μ.

Before we start a systematic analysis of these equations, it is worth comparing Eq. (3.225) to Eq. (3.132) and their integrability conditions which have the same structure except for the important detail of the dimensionality and signature. Therefore, we expect two main types of solutions: configurations with $U(1)$ holonomy on a 2-dimensional (spacelike) subspace and configurations with $U(1)$ holonomy in a null direction, which is the new possibility allowed by the Lorentzian signature. These expectations are also supported by the Fierz identities

$$\not{n}\epsilon = -i\epsilon\,, \tag{3.231}$$

$$\not{l}\epsilon^* = 0\,, \tag{3.232}$$

which are satisfied automatically here, but will be interpreted as projections.

We will call these two possibilities B and A respectively.

Killing equations for the vector bilinears and first consequences

We are now ready to derive equations involving the bilinears, in particular the vector bilinears which we construct with ϵ and the auxiliary spinor η introduced in Appendix B.1.2. First we deal with the equations that do not involve derivative of the spinors. Acting with $\bar{\epsilon}$ on Eq. (3.226) and with $\bar{\epsilon^*}\gamma^\mu$ on the complex conjugate of Eq. (3.228) we get

$$\phi^I F_{IJ}{}^+{}_{\mu\nu} l^\nu = 0\,, \tag{3.233}$$

$$\epsilon^{IJKL}\phi_J F_{KL}{}^+{}_{\mu\nu} l^\nu = 0\,. \tag{3.234}$$

Acting with $\bar{\epsilon^*}$ and $\bar{\eta^*}$ on Eq. (3.227) we get[9]

[9]The first of these equations had already been obtained in the general case Eq. (3.58).

$$l \cdot \partial\tau = 0\,, \tag{3.235}$$

$$m^* \cdot \partial\tau = 0\,. \tag{3.236}$$

Now, from Eqs. (3.225) and (3.229) we find

$$\nabla_\mu l_\nu = 0\,, \tag{3.237}$$

$$\tilde{\mathcal{D}}_\mu n_\nu = -a^*_\mu m_\nu - a_\mu m^*_\nu\,, \tag{3.238}$$

$$\tilde{\mathcal{D}}_\mu m_\nu = -a_\mu l_\nu\,. \tag{3.239}$$

Let us now find the simplest implications of these equations.

To start with,Eqs. (3.233) and (3.234), together, imply for nonvanishing ϕ_I[10]

$$F_{IJ}{}^{+}{}_{\mu\nu} l^\nu = 0\,. \tag{3.240}$$

Using Eq. (A.29), we see that the vector field strengths must take the form

$$F_{IJ}{}^{+} = \tfrac{1}{2}\mathcal{F}_{IJ}\, l \wedge m^*\,, \tag{3.241}$$

$$F_{IJ}{}^{-} = \tfrac{1}{2}\tilde{\mathcal{F}}_{IJ}\, l \wedge m\,, \tag{3.242}$$

where $\mathcal{F}_{IJ}$ is a skew-symmetric $SU(4)$ matrix of scalars to be determined and $\tilde{\mathcal{F}}_{IJ}$ is its $SU(4)$ dual.

This solves completely Eq. (3.228), as can be seen using the Fierz identity

$$l_\mu \gamma^{\mu\nu}\epsilon^* = 3l^\nu\epsilon^*\,, \tag{3.243}$$

and we can substitute Eq. (3.241) into Eq. (3.226) the only remaining equation in which vector field strengths occur. Using the Fierz identities

$$\not{l}\epsilon^* = 0\,, \tag{3.244}$$

$$\not{m}^*\epsilon^* = -i\epsilon\,, \tag{3.245}$$

[10]This equation also follows from the general result Eq. (3.65) for vanishing scalars M_{IJ}.

it takes the form

$$\tilde{\mathcal{D}}_\mu\phi_I - \tfrac{1}{4\sqrt{2}}\sqrt{\Im\mathrm{m}\,\tau}\mathcal{F}_{IJ}\phi^J l_\mu = 0\,, \tag{3.246}$$

from which we find

$$\mathcal{F}_{IJ}\phi^J = \frac{4\sqrt{2}}{\sqrt{\Im\mathrm{m}\,\tau}} n^\mu \tilde{\mathcal{D}}_\mu\phi_I\,. \tag{3.247}$$

On the other hand, from Eqs. (3.235) and (3.236) we find that

$$d\tau = A\hat{l} + B\hat{m}^*\,. \tag{3.248}$$

There are two cases to be considered here: case A ($B=0$) and case B ($B\neq 0$). In case B, we can write

$$d\tau = B\left(\hat{m}^* + \frac{A}{B}\,\hat{l}\right) = B\hat{m}^{*\prime}\,, \tag{3.249}$$

after a redefinition of the type Eqs. (B.49) and (B.50). All the equations that we have written so far are covariant with respect to this kind of transformations and we just have to add primes (which we suppress immediately afterwards) everywhere. Thus, the case B is equivalent to $A=0$ and we can always assume that either A or B is always zero. Since the connection Q depends on τ, the holonomy is different in these two cases. These are the two cases we mentioned at the end of the previous section and we will deal with them separately afterwards.

Equations of motion and integrability constraints

Although we have not yet discussed the form of the metric, we already have enough information to study the equations of motion and check whether they satisfy the integrability conditions Eqs. (3.48)-(3.50).

Using the results of the previous section, we can write the equations of motion in the form[11]

$$\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{E}^\rho{}_\rho = R_{\mu\nu} + \left[\frac{|A|^2}{2(\Im\mathrm{m}\,\tau)^2} + \tfrac{1}{16}\Im\mathrm{m}\,\tau\,\mathcal{F}^2\right] l_\mu l_\nu + \frac{|B|^2}{2(\Im\mathrm{m}\,\tau)^2} m_{(\mu} m^*_{\nu)}\,, \tag{3.250}$$

$$\mathcal{E} = \frac{1}{\Im\mathrm{m}\,\tau}\left[l^\mu\partial_\mu A^* - B^* l^\mu a_\mu + m^\mu\partial_\mu B^* + \tfrac{i}{4}\frac{|B|^2}{\Im\mathrm{m}\,\tau}\right]\,, \tag{3.251}$$

[11] We have ignored all the terms that contain products AB etc.

$$\hat{\mathcal{E}}_{IJ} - \bar{\tau}\hat{\mathcal{B}}_{IJ} = -i(\Im\mathrm{m}\,\tau)\,d(\mathcal{F}_{IJ}\hat{l}\wedge\hat{m}^*)\,. \tag{3.252}$$

Substituting into Eqs. (3.48)-(3.50) and operating, we get

$$R_{\mu\nu}l^\nu = 0\,, \tag{3.253}$$

$$R_{\mu\nu}m^\nu - \frac{|B|^2}{4(\Im\mathrm{m}\,\tau)^2}m_\mu = 0\,, \tag{3.254}$$

$$l^\mu\partial_\mu A^* - B^* l^\mu a_\mu + m^\mu\tilde{\mathcal{D}}_\mu B^* + \tfrac{i}{4}\frac{|B|^2}{\Im\mathrm{m}\,\tau} = 0\,, \tag{3.255}$$

$$B^*\mathcal{F}_{IJ}\phi^J = 0\,. \tag{3.256}$$

We do not have a metric yet, but we can find $R_{\mu\nu}l^\nu$ and $R_{\mu\nu}m^\nu$ from the integrability conditions of Eqs. (3.225) and (3.229). Commuting the derivative and projecting with gamma matrices and spinors in the usual way, it is easy to find from Eq. (3.225)

$$R_{\mu\nu}l^\nu = -2i(d\zeta)_{\mu\nu}l^\nu\,, \tag{3.257}$$

$$\begin{aligned} R_{\mu\nu}m^\nu &= +2i(d\zeta)_{\mu\nu}m^\nu - 2i(dQ)_{\mu\nu}m^\nu \\ &= +2i(d\zeta)_{\mu\nu}m^\nu + \frac{|B|^2}{4(\Im\mathrm{m}\,\tau)^2}m_\mu\,, \end{aligned} \tag{3.258}$$

and from Eq. (3.229)

$$\begin{aligned} R_{\mu\nu}m^\nu &= 2i(d\zeta)_{\mu\nu}m^\nu - 2i(dQ)_{\mu\nu}m^\nu + 2(da)_{\mu\nu}l^\nu \\ &= +2i(d\zeta)_{\mu\nu}m^\nu + \frac{|B|^2}{4(\Im\mathrm{m}\,\tau)^2}m_\mu + 2(da)_{\mu\nu}l^\nu\,, \end{aligned} \tag{3.259}$$

$$\begin{aligned} R_{\mu\nu}n^\nu &= 2i(d\zeta)_{\mu\nu}n^\nu - 2i(dQ)_{\mu\nu}n^\nu + 2(da)_{\mu\nu}m^{*\,\nu} \\ &= 2i(d\zeta)_{\mu\nu}n^\nu + 2(da)_{\mu\nu}m^{*\,\nu}\,. \end{aligned} \tag{3.260}$$

Comparing now these three sets of equations, we get

$$(d\zeta)_{\mu\nu}l^{\nu} = (d\zeta)_{\mu\nu}m^{\nu} = 0\,, \quad \Rightarrow d\zeta = 0\,, \quad \Rightarrow \zeta = d\alpha\,, \tag{3.261}$$

locally, and, eliminating ζ by a local phase redefinition,

$$(da)_{\mu\nu}l^{\nu} \;=\; 0\,, \tag{3.262}$$

$$(da)_{\mu\nu}m^{*\,\nu} \;=\; -\tfrac{1}{2}R_{\mu\nu}n^{\nu}\,, \tag{3.263}$$

which tell us that

$$da = -\tfrac{1}{2}R_{\bar{z}u}\hat{m}\wedge\hat{m}^{*} + \tfrac{1}{2}R_{uu}\hat{l}\wedge\hat{m} + C\hat{l}\wedge\hat{m}^{*}\,, \tag{3.264}$$

where C is a function to be chosen so as to make this equation (and, hence, Eq. (3.229)) integrable.

Once ζ has been eliminated, we can solve Eq. (3.247) of $\mathcal{F}_{IJ}$ as follows:

$$\mathcal{F}_{IJ} = \frac{8\sqrt{2}}{\sqrt{\Im\mathrm{m}\,\tau}}n^{\mu}(\partial_{\mu}\phi_{[I})\phi_{J]}\,. \tag{3.265}$$

Metric

At this point we need information about the exact form of the metric. The most important piece of information comes from the covariant constancy of the null vector l^{μ}. Metrics admitting a covariantly constant null vector are known as *pp*-wave metrics and were first described by Brinkmann in Refs. [94, 95]. Since l^{μ} is a Killing vector and $d\hat{l} = 0$ we can introduce the coordinates u and v

$$l_{\mu}dx^{\mu} \;\equiv\; du\,, \tag{3.266}$$

$$l^{\mu}\partial_{\mu} \;\equiv\; \frac{\partial}{\partial v}\,. \tag{3.267}$$

The previous results imply that all the objects we are dealing with $(\tau, \phi_I, \mathcal{F}_{IJ})$ are independent of v.

Using these coordinates, a 4-dimensional *pp*-wave metric takes the form[12]

[12]The components of the connection and the Ricci tensor of this metric can be found in Appendix C.2.

$$ds^2 = 2du(dv + Kdu + \omega) - 2e^{2U} dz d\bar{z}\,, \qquad \omega = \omega_z dz + \omega_{\bar{z}} d\bar{z}\,, \tag{3.268}$$

where all the functions in the metric are independent of v and where either K or the 1-form ω could, in principle, be removed by a coordinate transformation. In this case, however, we have to be very careful because we have already used part of the freedom we had to redefine the spinors, and, therefore, the null tetrad, and we have to check that the tetrad integrability equations (3.237)-(3.239) are satisfied by our choices of e^U, K and ω.

We are now ready to study and solve each case separately.

Case A

This is the $B = 0$ case. $d\tau = A\hat{l}$ implies that $\tau = \tau(u)$ and $A = \dot{\tau}$. The connection Q can be integrated

$$Q = d\beta(u)\,, \tag{3.269}$$

and can be eliminated from all the equations by absorbing a phase into the spinors:

$$e^{-i\beta}\epsilon = \epsilon'\,, \qquad e^{i\beta}\eta = \eta'\,, \tag{3.270}$$

and similarly on the null tetrad.

To fix the form of the metric, we study the antisymmetric part of Eq. (3.239)

$$d\hat{m} + \hat{a} \wedge \hat{l} = dU \wedge \hat{m} + \hat{a} \wedge \hat{l} = 0\,, \tag{3.271}$$

which implies that U only depends on u and

$$\hat{a} = \dot{U}\hat{m} + C\hat{l}\,, \tag{3.272}$$

where D is a function to be found. Substituting into the antisymmetric part of Eq. (3.238) we find

$$d\hat{n} + \hat{a}^* \wedge \hat{m} + \hat{a} \wedge \hat{m}^* = d\hat{n} + C^*\hat{l} \wedge \hat{m} + C\hat{l} \wedge \hat{m}^* = 0\,, \tag{3.273}$$

which is solved by

$$n = dv + Kdu\,, \qquad C^* = -e^{-U}\partial_z K\,. \tag{3.274}$$

Now, comparing Eq. (3.272) with Eq. (3.264) we find that $R_{uz}=0$ which implies (since $\omega=0$) that $\dot{U}=0$.

Finally, to ensure supersymmetry, the integrability conditions Eqs. (3.253)-(3.256) have to be satisfied, and, with constant U all of them are automatically satisfied.

It also follows form the previous equations that the ϕ_Is can only depend on u and $\mathcal{F}_{IJ}$ is given by

$$\mathcal{F}_{IJ} = \frac{8\sqrt{2}}{\sqrt{\Im\mathrm{m}\,\tau}}\dot{\phi}_{[I}\phi_{J]}\,. \tag{3.275}$$

Now, let us consider the equations of motion. The scalar, Maxwell and Bianchi equations are automatically satisfied and the Einstein equation can be solved by a K satisfying

$$2\partial_z\partial_{\bar{z}}K = \frac{|\dot{\tau}|^2}{(\Im\mathrm{m}\,\tau)^2} + \tfrac{1}{16}\Im\mathrm{m}\,\tau\,\mathcal{F}^2\,. \tag{3.276}$$

These solutions preserve generically 1/4 of the supersymmetries.

Case B

This is the $A=0$ case. If we choose $m^* = e^U d\bar{z}$, then $d\tau = Bm^*$ implies $\tau=\tau(\bar{z})$ and $Be^U = \partial_{\bar{z}}\tau$. Substituting the corresponding connection 1-form Q into Eq. (3.239) one finds

$$B^* = \frac{g(z,u)}{\sqrt{\Im\mathrm{m}\,\tau}}\,, \tag{3.277}$$

$$\hat{a} = -\partial_{\underline{u}}\ln g\,\hat{m} + D\hat{l}\,, \tag{3.278}$$

where g is a holomorphic function of z and D is a function to be determined. The first of these relations tells us that

$$\partial_z\bar{\tau} = \frac{e^U}{\sqrt{\Im\mathrm{m}\,\tau}}g(z,u)\,, \tag{3.279}$$

is a holomorphic function of z, independent of u, and taking the derivative of both sides with respect to $\bar{z}$ we get

$$\frac{e^U}{\sqrt{\Im\mathrm{m}\,\tau}} = f(u)\,, \qquad g(z,u) = \frac{h(z)}{f(u)}\,, \tag{3.280}$$

where $f(u)$ is a real function of u.

Substituting now $\hat{a}$ into the antisymmetric part of Eq. (3.238) we find that $\hat{n}$ is given by

$$\hat{n} = dv + \omega \,, \tag{3.281}$$

(so $K = 0$ in the metric Eq. (3.268)) where the 1-form ω satisfies

$$f_{z\bar{z}} = e^{2U}\partial_{\underline{u}} \ln (B/B^*) = 0 \,, \tag{3.282}$$

and D is given by

$$D^* = -\dot{\omega}_z e^{-U} \,. \tag{3.283}$$

Now that we have determined $\hat{a}$ we have to check that it satisfies the integrability condition Eq. (3.264). This requires the following equations to be satisfied:

$$R_{u\bar{z}} + \tfrac{i}{2}\frac{\partial_{\underline{u}} \ln fB}{\Im\mathrm{m}\,\tau} = 0 \,, \tag{3.284}$$

$$R_{uu} - [\partial^2_{\underline{u}} \ln f + \partial_{\underline{u}} \ln f \partial_{\underline{u}} \ln f] - 2e^{-U}\partial_z D = 0 \,, \tag{3.285}$$

$$C - e^{-U}\partial_{\bar{z}} D = 0 \,. \tag{3.286}$$

Comparing with the integrability conditions Eqs. (3.253)-(3.256), we conclude that f must be a constant that we normalize $f = 1$ and that ω must be exact, and we can eliminate it. Further, the ϕ_Is must be constant and the vector field strengths must vanish.

All the equations of motion are automatically satisfied in these conditions, and the solutions are the *stringy cosmic strings* of Ref. [92].

Our result differs from Tod's, who used τ and $\bar{\tau}$ as coordinates and found very similar solutions with nontrivial ω that depend in a very complicated way on a function $g(\tau, u)$ an its complex conjugate. This function could be eliminated by a coordinate change in which all the u dependence and the 1-form ω disappear, recovering the stringy cosmic string solutions.

4

Supersymmetry, attractors and cosmic censorship

In spite of the impressive progress made during the last few years in the study of supersymmetric black-hole solutions, there are important questions that remain unanswered or whose answer is unclear. For instance, we know how to construct many supersymmetric black-hole-type solutions, but many of them are singular. Some of these become regular when string corrections are taken into account and for all the regular black hole solutions we seem to have a String Theory model that accounts for its entropy. How are the other singular solutions to be understood? How can it be that they are supersymmetric and yet there is no String Theory model for them? Or, if there is, why are they singular?

The main goal of this chapter is to try to answer this question by giving a set of conditions that supersymmetric black-hole-type solutions must satisfy in order to be admissible in the context of $N = 2, d = 4$ supergravity coupled to vector supermultiplets. Admissible solutions will be regular and will describe one or several black holes in static equilibrium, even though the system may have a finite global angular momentum, as is for example the case in the solution constructed in Ref. [45]. Furthermore, we expect only admissible solutions to have a miscroscopic String Theory model. We will argue that the non-admissible solutions are, in general, not truly supersymmetric in the sense that will be explained later on and the conditions of admissibility can be seen as conditions for a solution to be everywhere supersymmetric. For instance: the Kerr-Newman solution with equal charge and mass, which is singular but nevertheless commonly believed to be supersymmetric, is non-admissible according to our criteria. We will show that it fails to be supersymmetric at the singularity, where the sources might be located. Equivalently we can say that the Kerr-Newman field with $M = |q|$ is caused by non-supersymmetric sources. This explains why it is not described by any supersymmetric String Theory model. We will also show that, generically, rotating sources are not allowed by supersymmetry and that regular, supersymmetric solutions with angular momentum are always composite objects made out of several static black holes in equilibrium. The angular momentum has its origin in the dipole momenta of the electromagnetic fields corresponding to the distribution of charged black holes. Something similar happens for scalar fields: supersymmetric configurations satisfying our conditions can have non-trivial scalar fields but cannot have sources.

In order to prove these results, we will make use of the explicit knowledge of the most general solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets, which have recently been classified in Ref. [3][1] . All the asymptotically flat supersymmetric black hole solutions seem to belong to the timelike class, and, although they coincide with the solutions found in Ref. [96], the general formalism will allow us to make further progress in their understanding. In particular, we will use the *Killing Spinor Identities* (KSIs) [6, 71], which can be understood as integrability conditions for the Killing spinor equations, in order to study supersymmetry at the singular points where the sources of these solutions should be located.

The final ingredient will be the attractor equations of $N = 2, d = 4$ supergravity [10, 97–99]: these provide us with information about the sources thought of as being placed at the attractor points. In fact, we will find interesting relations between KSIs and attractor equations, the former showing explicitly that

1. supersymmetry always requires the absence of the kind of scalar hair called *primary* in Ref. [100], and that

2. when the attractor equations are satisfied there are no sources whatsoever for scalar hair.[2]

These results can be viewed as an extension of those of Ref. [20] in which it was observed that supersymmetry seems to act as a cosmic censor for static black-hole-type configurations but not for the stationary ones, such as the Kerr-Newman $M = |q|$ solution.

We shall study how the KSIs constrain the possible sources and singularities of black-hole-type solutions and the interplay with the attractor equations in a general way. The main result will be the formulation of three conditions that express the existence of supersymmetry everywhere in the solutions, including, particularly, the locations of the sources. These conditions should ensure the regularity of the admissible solutions and we study in very close detail several examples.

[1] In this paper we will not consider the coupling to hypermultiplets. The classification of the supersymmetric solutions with both vector multiplets and hypermultiplets is considered in Ref. [4].

[2] If there is more than one basin of attraction, contrary to what is assumed in this article, this last conclusion might change due to the *area codes* [101].

4.1 Timelike BPS solutions of $N=2, d=4$ SUEGRA

It was recently shown in Ref. [3] that all the supersymmetric solutions in the timelike class of $N=2, d=4$ supergravity coupled to n vector multiplets[3] can be constructed by setting the $2\bar{n}=2(n+1)$ components of a real, symplectic vector $\mathcal{I}=(\mathcal{I}^\Lambda, \mathcal{I}_\Lambda)$ equal to $2\bar{n}=2(n+1)$ real functions harmonic on 3-dimensional Euclidean space[4]

$$\mathcal{I} \equiv \begin{pmatrix} \mathcal{I}^\Lambda \\ \mathcal{I}_\Lambda \end{pmatrix}, \qquad \partial_m\partial_m\mathcal{I}^\Lambda = \partial_m\partial_m\mathcal{I}_\Lambda = 0\,, \qquad \Lambda = 0,1,\cdots,n\,. \tag{4.1}$$

This real section $\mathcal{I}$ enters the theory as the imaginary part of the section $\mathcal{V}/X$, where $\mathcal{V}$ is the covariantly-holomorphic canonical section defining special geometry:

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{pmatrix} \rightarrow \left\{ \begin{array}{lcl} \langle \mathcal{V} \mid \mathcal{V}^* \rangle & \equiv & \mathcal{L}^{*\Lambda}\mathcal{M}_\Lambda - \mathcal{L}^\Lambda\mathcal{M}^*_\Lambda = -i\,, \\ \mathfrak{D}_{i^*}\mathcal{V} & = & (\partial_{i^*} - \tfrac{1}{2}\partial_{i^*}\mathcal{K})\mathcal{V} = 0\,, \\ \langle \mathfrak{D}_i\mathcal{V} \mid \mathcal{V} \rangle & = & 0\,. \end{array} \right. \tag{4.2}$$

X on the other hand is proportional to the complex, scalar bilinear constructed out of the Killing spinors: supersymmetry and consistency of the solutions imply that it can be expressed in terms of $\mathcal{I}$, see e.g. Ref. [3] or Eq. (4.7).

Eqs. (4.1) are sometimes known as the *generalized stabilization equations*, the standard stabilization equations having the same form but with the harmonic functions $(\mathcal{I}^\Lambda, \mathcal{I}_\Lambda)$ replaced by magnetic and electric charges, *e.g.* (p^Λ, q_Λ).

The real part of $\mathcal{V}/X$, denoted by $\mathcal{R} \equiv (\mathcal{R}^\Lambda, \mathcal{R}_\Lambda)$ can, in principle, be written in terms of the real harmonic functions, which is usually referred to as "solving the stabilization equations". In theories with a prepotential, the homogeneity properties of the prepotential allow us to write

$$\mathcal{M}_\Lambda/X = \frac{\partial\mathcal{F}(\mathcal{L}^\cdot/X)}{\partial(\mathcal{L}^\Lambda/X)}\,. \tag{4.3}$$

Taking the imaginary part of this equation, we have

$$\mathcal{I}_\Lambda(\mathcal{R}^\cdot, \mathcal{I}^\cdot) = \mathcal{I}_\Lambda\,, \tag{4.4}$$

[3] These solutions were first found in slightly different form in Ref. [96] and the procedure followed in Ref [3] shows that they are the only solutions in this class.

[4] If the functions are not harmonic, the field configurations are still supersymmetric, but are *not* solutions of the equations of motion.

which implicitly defines $\mathcal{R}^{\Lambda}(\mathcal{I}^{\cdot},\mathcal{I}_{\cdot})$, although solving these equations can be extremely hard and in general the explicit solution is unknown.

The real part of Eqs. (4.3) and the above solutions give straightforwardly the functions $R_{\Lambda}(R^{\cdot}(\mathcal{I}^{\cdot},\mathcal{I}_{\cdot}),\mathcal{I}^{\cdot})$.

Having the complete symplectic section $\mathcal{V}/X$ entirely given in terms of the real harmonic functions, one can construct the fields of the solutions as follows:

1. The n complex scalar fields Z^i are given by the quotients

$$Z^i = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,. \tag{4.5}$$

2. The metric has the form

$$ds^2 = 2|X|^2(dt+\omega)^2 - \frac{1}{2|X|^2}dx^i dx^i\,, \qquad i,j = 1,2,3\,, \tag{4.6}$$

   where

$$\frac{1}{2|X|^2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle\,, \tag{4.7}$$

   and ω is a time-independent 1-form on Euclidean 3-dimensional space satisfying the equation

$$(d\omega)_{mn} = 2\epsilon_{mnp}\langle\, \mathcal{I} \mid \partial_p \mathcal{I}\,\rangle\,. \tag{4.8}$$

3. The symplectic vector of field strengths and their duals $F = (F^{\Lambda}, \tilde{F}_{\Lambda})$ is given by

$$F = -\tfrac{1}{2}\{d[\mathcal{R}\hat{V}] - {}^{\star}[d\mathcal{I}\wedge\hat{V}]\}\,, \qquad \hat{V} = 2\sqrt{2}|X|^2(dt+\omega)\,. \tag{4.9}$$

The Killing spinors of these solutions have the form

$$\epsilon_I = X^{1/2}\epsilon_{I\,0}\,, \qquad \partial_\mu \epsilon_{I\,0} = 0\,, \qquad \epsilon_{I\,0} + i\gamma_0\epsilon_{IJ}\epsilon^{J}{}_{0} = 0\,, \tag{4.10}$$

which implies

$$\epsilon_I + i\gamma_0 e^{i\alpha}\epsilon_{IJ}\epsilon^J = 0\,, \qquad e^{i\alpha} = (X/X^*)^{1/2}\,. \tag{4.11}$$

Observe that we can write

$$X = \frac{\mathcal{L}^\Lambda(Z, Z^*)}{\mathcal{R}^\Lambda + i\mathcal{I}^\Lambda}\,, \tag{4.12}$$

for any Λ.

4.1.1 Killing Spinor Identities

All supersymmetric configurations satisfy the *Killing Spinor Identities* relating the Einstein equations $\mathcal{E}^{\mu\nu}$, the Maxwell equations $\mathcal{E}_\Lambda{}^\mu$, the Bianchi identities $\mathcal{B}^{\Lambda\,\mu}$ and the scalar equations of motion $\mathcal{E}^i$ [3, 6, 71]

$$\mathcal{E}_a{}^\mu \gamma^a \epsilon_I - 4i\langle\, \mathcal{E}^\mu \mid \mathcal{V}\,\rangle \epsilon_{IJ}\epsilon^J \;=\; 0\,, \tag{4.13}$$

$$\mathcal{E}^i \epsilon^I + 2i\langle\, \not{\mathcal{E}} \mid \mathcal{U}^{*\,i}\,\rangle \epsilon^{IJ}\epsilon_J \;=\; 0\,, \tag{4.14}$$

where $\mathcal{E}^\mu$ is the symplectic vector $(\mathcal{B}^{\Lambda\,\mu}, \mathcal{E}_\Lambda{}^\mu)$.

In the timelike case, they lead to the following identities in an orthonormal frame

$$\mathcal{E}^{ab} \;=\; \eta^a{}_0 \eta^b{}_0 \mathcal{E}^{00}\,, \tag{4.15}$$

$$\langle\, \mathcal{V}/X \mid \mathcal{E}^a\,\rangle \;=\; \tfrac{1}{4}|X|^{-1}\mathcal{E}^{00}\delta^a{}_0\,, \tag{4.16}$$

$$\langle\, \mathcal{U}^*_{i^*} \mid \mathcal{E}^a\,\rangle \;=\; \tfrac{1}{2}e^{-i\alpha}\mathcal{E}_{i^*}\delta^a{}_0\,. \tag{4.17}$$

These equations imply directly

$$\mathcal{E}^{0m} = 0\,, \quad \mathcal{E}^{mn} = 0\,, \quad \langle\, \mathcal{V} \mid \mathcal{E}^m\,\rangle = 0\,, \quad \langle\, \mathcal{U}^*_{i^*} \mid \mathcal{E}^m\,\rangle = 0\,. \tag{4.18}$$

Further, the r.h.s. of Eq. (4.15) is real, and this leads to two important identities:

$$\langle\, \mathcal{I} \mid \mathcal{E}^0\,\rangle \;=\; 0\,, \tag{4.19}$$

$$\mathcal{E}^{00} \;=\; \pm 4|\langle\, \mathcal{V} \mid \mathcal{E}^0\,\rangle|\,. \tag{4.20}$$

4.1.2 Attractor equations

It is well-known that, in general, the scalar fields of the black-hole solutions of these theories have certain attractor values that depend solely on the electric and magnetic charges and which are attained at the event horizons irrespectively of the chosen asymptotic values [97, 98].[5] The attractor values are those which extremize a specific function; furthermore, the absolute value squared of the central charge for the attractor values is essentially the horizon area [10, 99]. Here we are going to rederive these results using our notation and to relate them to the KSIs. We also want to improve the previous derivations by making explicit use of the knowledge of all the supersymmetric configurations.

Let us consider single, static, asymptotically flat, spherically symmetric, black-hole-type solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets: they are given by real harmonic functions of the form

$$\mathcal{I} = \mathcal{I}_\infty + \frac{q}{r}\,, \tag{4.21}$$

which is the general choice compatible with the assumptions. The metric can be conveniently written in spherical coordinates as

$$ds^2 = 2|X|^2 dt^2 - \frac{1}{2|X|^2}[dr^2 + r^2 d\Omega^2_{(2)}]\,. \tag{4.22}$$

This metric describes black holes if

$$-g_{rr} = \frac{1}{2|X|^2} \stackrel{r\to\infty}{\longrightarrow} 1 + \frac{2M}{r}\,, \tag{4.23}$$

is always finite for finite r, whence M, which is the mass, must be positive. Further, we have to require

$$\frac{1}{2|X|^2} \stackrel{r\to 0}{\longrightarrow} \frac{A}{4\pi r^2} > 0\,, \tag{4.24}$$

which imposes the existence of an event horizon with area $A > 0$ at $r = 0$ instead of a naked singularity.

The existence of attractors (fixed points) of the scalar fields follows from the fact that in supersymmetric configurations, the scalars satisfy first-order differential equations, as follows immediately from the Killing spinor equations associated to the gaugino supersymmetry transformation rule:

[5] If there are multiple attractor regions, it might happen that there is some residual dependency on the asymptotic values. Here we assume there to be only one attractor region.

$$\delta_\epsilon \lambda^{Ii} = i \not{\partial} Z^i \epsilon^I + \epsilon^{IJ} \not{G}^{i\,+} \epsilon_J = 0\,. \tag{4.25}$$

To derive the needed first-order equations, we first use the time-independence of the solutions

$$i\gamma^m \partial_m Z^i \epsilon^I - 4\epsilon^{IJ} G^{i\,+}{}_{0m} \gamma^m \gamma^0 \epsilon_J = 0\,, \tag{4.26}$$

and then the known constraint Eq. (4.11) as to obtain

$$(\partial_m Z^i - 4e^{i\alpha} G^{i\,+}{}_{0m})\gamma^m \epsilon^I = 0\,, \quad \Rightarrow \partial_m Z^i = 4e^{i\alpha} G^{i\,+}{}_{0m}\,. \tag{4.27}$$

Going over to curved indices, the equation takes the form

$$\frac{dZ^i}{dr} = 2\sqrt{2} G^{i\,+}{}_{tr}/X^*\,. \tag{4.28}$$

The self-duality of $G^{i\,+}$ allows us to express the $G^{i\,+}{}_{tr}$ component in terms of the $G^{i\,+}{}_{\theta\phi}$:

$$G^{i\,+}{}_{tr} = i({}^\star G^{i\,+})_{\theta\phi} = -i\frac{2|X|^2}{r^2 \sin\theta} G^{i\,+}{}_{\theta\phi}\,, \tag{4.29}$$

which combined with

$$G^{i\,+} = \mathcal{T}^i{}_\Lambda F^{\Lambda\,+} = \tfrac{i}{2}\mathcal{G}^{ij^*}\langle\, \mathfrak{D}_{j^*}\mathcal{V}^* \mid F\,\rangle = \tfrac{i}{2}\mathcal{G}^{ij^*}\mathfrak{D}_{j^*}\langle\, \mathcal{V}^* \mid F\,\rangle\,, \tag{4.30}$$

leads to

$$\frac{dZ^i}{dr} = 2\sqrt{2}\frac{X}{r^2\sin\theta}\mathcal{G}^{ij^*}\mathfrak{D}_{j^*}\langle\, \mathcal{V}^* \mid F_{\theta\phi}\,\rangle\,. \tag{4.31}$$

Since the form of all the fields in terms of $\mathcal{I}(r)$ is in principle known, we can try to find a more explicit form for this equation: using the general form of the vector fields Eq. (4.9) and of $\mathcal{I}(r)$, Eq. (4.21), we find

$$F_{\theta\phi} = \tfrac{1}{\sqrt{2}} r^2 \sin\theta \frac{d\mathcal{I}}{dr} = -\frac{q}{\sqrt{2}}\sin\theta\,. \tag{4.32}$$

After substituting this into Eq. (4.31), one ends up with

$$\frac{dZ^i}{d\rho} = 2X\mathcal{G}^{ij^*}\mathfrak{D}_{j^*}\mathcal{Z}^*\,, \tag{4.33}$$

where $\rho \equiv 1/r$ and where

$$\mathcal{Z}(Z,q) \equiv \langle\, \mathcal{V} \mid q \,\rangle\,, \tag{4.34}$$

is the *central charge of the theory* [102]. Observe that the presence of the factor X in the r.h.s. is crucial for it to have zero global Kähler weight, just as the l.h.s. Further observe that the r-dependence is only through the scalars $Z^i(r)$!

The r.h.s. of this system of differential equations depends only on the scalar fields Z^i, and, thus, it is an autonomous system of ordinary differential equations[6] that has fixed points Z^i_{fix} at the values at which the r.h.s. vanishes

$$\mathfrak{D}_i \mathcal{Z}|_{Z^i = Z^i_{\text{fix}}} = 0\,. \tag{4.35}$$

If the solution of this system of equations exists, it gives the fixed values of the scalars Z^i_{fix} as functions of the electric and magnetic charges only

$$Z^i_{\text{fix}} = Z^i_{\text{fix}}(q)\,, \tag{4.36}$$

since the asymptotic values (moduli) Z^i_∞ do not occur in the above differential equation. The fixed values are reached by the scalars at the value $\rho = \infty$, i.e. $r = 0$, which is where the event horizon would be, as discussed at the beginning of this section and in what follows.

The fixed values may or may not be admissible, i.e. they may or may not belong to the definition domain of the complex coordinates Z^i. If the asymptotic values Z^i_∞ are admissible and the fixed values $Z^i_{\text{fix}}(q)$ are not, there must be a singularity between $r = \infty$ and $r = 0$, which will induce a curvature singularity. We will require both the asymptotic and the fixed values to be admissible. These aspects will be discussed in Section 4.2.

Black-hole solutions whose scalars take the asymptotic values $Z^i_\infty = Z^i_{\text{fix}}$ have constant scalar fields, and are called *doubly extreme black holes*. These values are the ones that extremize, not the central charge, but the zero-Kähler-weight combination $e^{\mathcal{K}/2}\mathcal{Z}$:

$$\mathfrak{D}_i \mathcal{Z}|_{Z^i = Z^i_{\text{fix}}} = e^{-\mathcal{K}/2}\partial_i \left(e^{\mathcal{K}/2}\mathcal{Z}\right)\big|_{Z^i = Z^i_{\text{fix}}} = 0\,. \tag{4.37}$$

[6] The use of the variable $\rho = 1/r$ is essential in this argument. it is easy to see that the derivatives of the scalar fields of typical black-hole solutions w.r.t. to r do not vanish at $r = 0$, while their derivatives w.r.t. ρ do..

Consequences of the existence of attractors

There are no more scalar fields in the theory, but in the timelike supersymmetric solutions there is another scalar object[7] that satisfies a first-order differential equation: X. From the Killing spinor equation associated to the gravitino supersymmetry transformation rule it is possible to derive [3]

$$\mathfrak{D}_\mu X = -iT^+{}_{\mu\nu}V^\nu\,, \tag{4.38}$$

where V^μ is the timelike Killing vector constructed from the Killing spinor. The graviphoton field strength can be written in the form

$$T^+ = \langle\, \mathcal{V} \mid F \,\rangle\,, \tag{4.39}$$

and, together with

$$V^\nu F_{\nu\mu} = 2\nabla_\mu(|X|^2\mathcal{R})\,, \tag{4.40}$$

the equation for X becomes

$$\mathfrak{D}_\mu X = 2i\langle\, \mathcal{V} \mid \nabla_\mu(|X|^2\mathcal{R}) \,\rangle\,. \tag{4.41}$$

Dividing both sides by X and expanding the r.h.s. using $\mathcal{V}/X = \mathcal{R} + i\mathcal{I}$ we get

$$\frac{\mathfrak{D}_\mu X}{X} = 2i|X|^2\langle\, \mathcal{R} \mid \nabla_\mu\mathcal{R} \,\rangle - 2\nabla_\mu|X|^2\langle\, \mathcal{I} \mid \mathcal{R} \,\rangle - 2|X|^2\langle\, \mathcal{I} \mid \nabla_\mu\mathcal{R} \,\rangle\,. \tag{4.42}$$

Now, from Eq. (4.7)

$$-2\langle\, \mathcal{I} \mid \nabla_\mu\mathcal{R} \,\rangle = \nabla_\mu|X|^{-2} - 2\langle\, \mathcal{R} \mid \nabla_\mu\mathcal{I} \,\rangle\,, \tag{4.43}$$

and we get

$$\frac{\mathfrak{D}_\mu X}{X} = 2i|X|^2\langle\, \mathcal{R} \mid \nabla_\mu\mathcal{R} \,\rangle - 2|X|^2\langle\, \mathcal{R} \mid \nabla_\mu\mathcal{I} \,\rangle\,. \tag{4.44}$$

Finally, using

$$\langle\, \mathcal{R} \mid \nabla_\mu\mathcal{R} \,\rangle = \langle\, \mathcal{I} \mid \nabla_\mu\mathcal{I} \,\rangle\,, \tag{4.45}$$

[7] In previous derivations in the literature the absolute value $|X| = e^U$ is considered, but then the Kähler weights and the reality properties of the two sides of the equations derived are different.

which is proved in Appendix F, we arrive at[8]

$$\mathfrak{D}_\mu X^{-1} = 2\langle\, \mathcal{V}^* \mid \nabla_\mu \mathcal{I}\,\rangle\,. \tag{4.47}$$

This equation is valid for all supersymmetric configurations in the timelike class. For those considered in this section we arrive at the equation we were looking for:

$$\mathfrak{D}_\rho X^{-1} = 2\mathcal{Z}^*\,. \tag{4.48}$$

The real and imaginary parts of this equation are

$$\frac{d(-g_{rr})}{d\rho} \;=\; 2\Re\mathrm{e}(\mathcal{Z}^*/X^*) = 2\langle\, \mathcal{R} \mid q\,\rangle\,, \tag{4.49}$$

$$\frac{d\alpha}{d\rho} + \mathcal{Q}_\rho \;=\; |X|^2 - 2\Im\mathrm{m}(\mathcal{Z}^*/X^*) = 2\langle\, \mathcal{I} \mid q\,\rangle = 2\langle\, \mathcal{I}_\infty \mid q\,\rangle\,. \tag{4.50}$$

For the spherically symmetric solutions under consideration ω vanishes and this requires the phase of X to be covariantly constant, i.e.

$$\langle\, \mathcal{I} \mid q\,\rangle = \langle\, \mathcal{I}_\infty \mid q\,\rangle = 0\,. \tag{4.51}$$

We will later show that this is equivalent to the requirement that the NUT charge vanishes. Since there is only dependence on ρ, the phase of X can simply be gauged away by means of a Kähler transformations. The phase of $\mathcal{Z}$ is then also constant, whence $\mathcal{Z}/X$ is real, which can be used to write

$$\frac{d|X|^{-1}}{d\rho} = \pm 2|\mathcal{Z}|\,. \tag{4.52}$$

The $\pm$ sign is the sign of $\langle\, \mathcal{R} \mid q\,\rangle$ and we can argue that it has to be positive if the mass is going to be positive: if we take Eq. (4.49) at $\rho = 0$ ($r = \infty$), we find that the mass of the solution is given by the linear combination of charges and moduli

$$M = \langle\, \mathcal{R}_\infty \mid q\,\rangle\,. \tag{4.53}$$

Observe that there is no *a priori* guarantee that $M > 0$: this is a condition that has to be imposed independently as to avoid singularities. We will do so and will

[8]Observe that the compatibility between Eq. (4.7) and the following equations requires the identity

$$\langle\, \nabla_\mu \mathcal{R} \mid \mathcal{I}\,\rangle = \langle\, \mathcal{R} \mid \nabla_\mu \mathcal{I}\,\rangle\,, \tag{4.46}$$

to hold. For theories admitting a prepotential, this is done in Appendix F.

only consider the positive sign above; Eq. (4.52) is then the expression found in the literature.

If we take another derivative of Eq. (4.49) and use Eq. (4.52), we find

$$\frac{d^2(-g_{rr})}{d\rho^2} = 2\frac{d|X^{-1}|}{d\rho}|\mathcal{Z}| + 2|X|^{-1}\frac{d|\mathcal{Z}|}{d\rho} = 4|\mathcal{Z}|^2 + 2|X|^{-1}\left(\frac{dZ^i}{d\rho}\partial_i|\mathcal{Z}| + \text{c.c.}\right) . \quad (4.54)$$

Now, at $\rho = \rho_{\text{fix}} = 0$ we have $Z^i = Z_{\text{fix}}$ and $dZ^i/d\rho = 0$, and the above equation takes on the form

$$\frac{A}{2\pi} = 4|\mathcal{Z}_{\text{fix}}|^2 . \quad (4.55)$$

Again, there is no *a priori* guarantee that $|\mathcal{Z}_{\text{fix}}| \neq 0$, which therefore is another condition that has to be imposed independently as to avoid singularities. Actually, even though in this expression A is basically an absolute value, the positivity of A is only guaranteed if the scalar fields take admissible values, the mass is positive etc.

These identities allow us to find two interesting expressions for $|\mathcal{Z}_{\text{fix}}|$. Expanding the two sides of Eq. (4.49) as a power series in ρ we find

$$\frac{A}{2\pi} = 2\langle \left.\frac{d\mathcal{R}}{d\rho}\right|_{\rho=0} \mid q \rangle . \quad (4.56)$$

Using the expressions in Appendix F we get [99, 103]

$$|\mathcal{Z}_{\text{fix}}|^2 = \tfrac{1}{2}\langle \left.\frac{d\mathcal{R}}{d\rho}\right|_{\rho=0} \mid q \rangle = -\tfrac{1}{2}q^T\mathcal{M}(\mathcal{F}_{\text{fix}})q , \quad (4.57)$$

where

$$\mathcal{M}(\mathcal{F}) \equiv \begin{pmatrix} \Im\text{m}\mathcal{F} + \Re\text{e}\mathcal{F}\Im\text{m}\mathcal{F}^{-1}\Re\text{e}\mathcal{F} & -\Im\text{m}\mathcal{F}^{-1}\Re\text{e}\mathcal{F} \\ & \\ -\Re\text{e}\mathcal{F}\Im\text{m}\mathcal{F}^{-1} & \Im\text{m}\mathcal{F}^{-1} \end{pmatrix} . \quad (4.58)$$

A direct computation of $|\mathcal{Z}_{\text{fix}}|^2$ gives

$$|\mathcal{Z}_{\text{fix}}|^2 = |\langle \mathcal{V}_{\text{fix}} \mid q \rangle|^2 = -\langle q \mid \mathcal{V}_{\text{fix}} \rangle\langle \mathcal{V}^*_{\text{fix}} \mid q \rangle . \quad (4.59)$$

The matrix of this bilinear is

$$\mid \mathcal{V}_{\text{fix}} \rangle\langle \mathcal{V}^*_{\text{fix}} \mid = -\begin{pmatrix} \mathcal{M}_\Lambda\mathcal{M}^*_\Sigma & -\mathcal{M}_\Lambda\mathcal{L}^{*\,\Sigma} \\ & \\ -\mathcal{L}^\Lambda\mathcal{M}^*_\Sigma & \mathcal{L}^\Lambda\mathcal{L}^{*\,\Sigma} \end{pmatrix}_{\text{fix}} . \quad (4.60)$$

We can use the relation

$$\mathcal{L}^{*\Lambda}\mathcal{L}^{\Sigma} = -\tfrac{1}{2}\Im\mathrm{m}(\mathcal{N})^{-1|\Lambda\Sigma} - f^{\Lambda}{}_{i}\mathcal{G}^{ii^*}f^{*\Sigma}{}_{i^*}\,, \tag{4.61}$$

taking into account that at the fixed point the second term in the r.h.s. will not contribute, and that only its symmetric part will contribute, to get [99, 103]

$$|\mathcal{Z}_{\mathrm{fix}}|^2 = -\tfrac{1}{2}q^T\mathcal{M}(\mathcal{N}_{\mathrm{fix}})q\,. \tag{4.62}$$

So far we have checked that the coefficient of the ρ^2 term of $-g_{rr}$ is given by the value of the central charge at the fixed point but, if there are terms of higher order in ρ in $-g_{rr}$ there will not be a regular horizon. We can, however, see that taking another derivative of $-g_{rr}$ w.r.t. ρ at $\rho = 0$ will give zero if the attractor equations (4.35) are satisfied and the same will happen for higher derivatives.

Summarizing we can say that the attractor equations (plus the positivity of the mass, which is not guaranteed) seem to be sufficient conditions to have regular, static, spherically symmetric black holes.

Finally, observe that Eq. (4.53) plus the identification, which will be established later on, between the NUT charge and the linear expression of the charges

$$N = \langle\, \mathcal{I}_\infty \mid q\,\rangle\,, \tag{4.63}$$

lead to a complex BPS relation

$$M + iN = \langle\, (\mathcal{V}/X)_\infty \mid q\,\rangle\,. \tag{4.64}$$

We will argue that supersymmetry requires N to vanish, whence the above relation reads

$$M = \pm\sqrt{2}|\mathcal{Z}_\infty|\,, \tag{4.65}$$

which is the standard BPS relation between mass and central charge. Of course, only the positive sign will be admissible.

4.2 Relations between the $N = 2, d = 4$ KSIs, attractors and sources

The equations of motion[9] for supersymmetric configurations of supergravity theories satisfy certain relations known as *Killing spinor identities* (*KSI*s), which can also be derived from the integrability conditions of the Killing spinor equations [6, 71]. We have unbroken supersymmetry wherever the Killing spinors exist, and these exist, locally, wherever the KSIs are satisfied. Thus, if we are to have unbroken supersymmetry everywhere we must demand the KSIs to be satisfied everywhere. In this section we are going to study the consequences of demanding the black-hole solutions of $N = 2, d = 4$ supergravity to be everywhere supersymmetric.

The KSIs of $N = 2, d = 4$ supergravity are given in Eqs. (4.13) and (4.14) and they lead to Eqs. (4.15)-(4.20) for configurations in the timelike class. Since we are going to consider configurations that solve the equations of motion, it may seem that the KSIs are automatically satisfied. However, most solutions have singularities at which the equations of motion are not satisfied, i.e. one has $\mathcal{E}(\phi) = \mathcal{J}(\phi)$. The r.h.s. of the equations of motion at the singularities can be associated to sources for the corresponding fields and the KSIs are then understood as relations between the possible sources of supersymmetric solutions: the KSIs put constraints on possible sources of supersymmetric solutions.

Let us consider from this point of view the KSIs Eqs. (4.15)-(4.20): the first of them, Eq. (4.15), tells us that the components $\mathcal{E}^{0m}$ and $\mathcal{E}^{mn}$ of the Einstein equations must vanish automatically for supersymmetric configurations and they must do so everywhere if the solutions are everywhere supersymmetric. This means that the sources $\mathcal{J}^{0m}$ and $\mathcal{J}^{mn}$ of the Einstein equation must vanish identically everywhere

$$\mathcal{J}^{0m} = \mathcal{J}^{mn} = 0\,. \tag{4.66}$$

Hence, singular (delta-like) sources are not allowed, and in particular this means that no localized sources of angular momentum are allowed.

Any singular contributions to $\mathcal{J}^{0m}$ and $\mathcal{J}^{mn}$ must originate in the R^{0m} components of the Ricci tensor; more precisely, they come from the term $\partial_{\underline{m}}(d\omega)_{\underline{mn}}$, where ω is the 1-form that appears off-diagonally in the metric of the timelike supersymmetric solutions of $N = 2, d = 4$ supergravity Eq. (4.6). Therefore, using Eq. (4.8) and defining the complex 3-dimensional vector $\vec{\mathcal{W}}$

$$\vec{\mathcal{W}} = (\mathcal{W}_{\underline{m}}) \equiv (\langle\, \mathcal{V}/X \mid \partial_{\underline{m}}\mathcal{I}\,\rangle)\,, \qquad \Im\mathrm{m}\,(\mathcal{W}_{\underline{m}}) = \tfrac{1}{4}\epsilon_{mnp}(d\omega)_{\underline{np}} = \langle\, \mathcal{I} \mid \partial_{\underline{m}}\mathcal{I}\,\rangle\,, \tag{4.67}$$

[9] By *equation of motion* $\mathcal{E}(\phi)$ of a given field ϕ we will mean here the l.h.s. of the equation of motion $\delta S/\delta\phi = \mathcal{E}(\phi) = 0$. This slight abuse of language should lead to no confusions.

we can translate the above KSIs, Eqs. (4.66), to the condition

$$\Im\mathrm{m}\,(\vec{\nabla}\times\vec{\mathcal{W}})=0\,,\tag{4.68}$$

which has to be imposed everywhere. Actually, only the singular parts of this equation have to be taken into account since, dealing with solutions, the finite parts must be canceled in the equations of motion by other finite contributions. Therefore, from now on we will ignore all finite contributions to this equation.

Let us consider the real and imaginary parts of Eq. (4.16), namely Eq. (4.20) and (4.19). The real part gives us two important pieces of information: first, it tells us that the component $\mathcal{J}^{00}$ of the source of the Einstein equation is related to component $\mathcal{J}^{0}$ of the source of the combined Maxwell and Bianchi equations $\mathcal{E}^{a}$. If the electromagnetic fields have only one static point-like source at $r=0$, $\mathcal{E}^{t}\sim\frac{1}{\sqrt{2}}q\delta^{(3)}(\vec{x})/\sqrt{|g|}$, then using the fact that $\mathcal{Z}/X$ is real (see Eq. (4.51) and the previous discussion)

$$\mathcal{E}^{0t}=\pm 2\sqrt{2}\,|\mathcal{Z}|\big|_{r=0}\,\delta^{(3)}(\vec{x})/\sqrt{|g|}\,,\tag{4.69}$$

which shows that, if the attractor equations are satisfied, the source for the Einstein equations is just $\pm|\mathcal{Z}_{\text{fix}}(q)|$. The sign is related to the positivity of $\langle\,\mathcal{R}\mid q\,\rangle$, which is, as was discussed before, associated to the positivity of the mass etc. This is the only value admissible by supersymmetry, since we can understand this source as a source of energy. However, if the scalars take non-admissible values we will find the wrong sign or a zero at $r=0$ and supersymmetry will be broken at the source: we will have to require that the attractor equations are solved by admissible values of the scalars.

The second piece of information we can obtain from the real part concerns the spacelike components of the electromagnetic sources. Combined with the spacelike components of the imaginary part, Eq. (4.19), we get the condition

$$\langle\,\mathcal{V}/X\mid\mathcal{J}^{\underline{m}}\,\rangle=0\,.\tag{4.70}$$

Let us now consider the time component of the imaginary part of the KSI Eq. (4.16), Eq. (4.19):

$$\langle\,\mathcal{I}\mid\mathcal{J}^{t}\,\rangle=0\,.\tag{4.71}$$

To find the physical meaning of this condition we use the explicit form of the symplectic vector of vector field strengths F for timelike BPS solutions Eq. (4.9):

$$\mathcal{J}^{\mu}=\mathcal{E}^{\mu}=-({}^{\star}dF)^{\mu}=|X|^{2}\,(\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I})\,V^{\mu}=\frac{\delta^{\mu}{}_{t}}{\sqrt{2}}\frac{\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}}{\sqrt{|g|}}\,.\tag{4.72}$$

This result tells us that the KSIs Eq. (4.70) are always satisfied and that the KSI Eq. (4.71) is equivalent to the condition

$$\langle \mathcal{I} \mid \partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I} \rangle = \Im\mathrm{m}\,(\partial_{\underline{m}}\mathcal{W}_{\underline{m}}) = 0\,, \tag{4.73}$$

which is nothing but the integrability condition for the equation determining ω, which now has to be satisfied everywhere as a consequence of demanding unbroken supersymmetry *everywhere*. For the point-like sources considered above, these equations take the form

$$\sum_A \langle \mathcal{I} \mid q_A \rangle \delta^{(3)}(\vec{x} - \vec{x}^A)/\sqrt{|g|} = 0\,. \tag{4.74}$$

The consequences of imposing this condition were first studied by Denef and Bates in Refs. [75, 76] in the context of general $N = 2, d = 4$ supergravity, but was studied earlier by Hartle and Hawking in Ref. [104] in the context of Israel-Wilson-Perjés (IWP) solutions of the Einstein-Maxwell theory. As shown by Tod in Ref. [11] these are precisely the timelike solutions of pure $N = 2, d = 4$ supergravity and a special case of the general problem that we are going to study. Hartle and Hawking were motivated, not by supersymmetry, but rather by the prospect of finding regular solutions describing more than one black hole. They were, in particular, worried about possible string singularities related to NUT charges. These singularities can be eliminated by compactifying the time coordinate with certain period [105], but at the price of losing asymptotic flatness. Let us consider a possible string singularity parametrized by z and choose polar coordinates ρ, ϕ around it. If one considers the integral of the 1-form ω that appears in the metric along a loop of radius R enclosing the possible string singularity at two different points z_1 and z_2, denoted by $I(R, z_{1,2})$, one can use Stokes' theorem to derive

$$I(R, z_1) - I(R, z_2) = \int_{\Sigma^2} d\omega = 2\int_{\Sigma^2} \star_3 \Im\mathrm{m}\mathcal{W}\,, \tag{4.75}$$

where Σ^2 is a surfaces whose boundaries are the loops of radius R at $z_{1,2}$. In the zero radius limit Σ^2 is a closed surface that crosses the possible string singularity at z_1 and z_2 and we have

$$\begin{aligned} 2\pi \lim_{R\to 0} R[\omega_\phi(R, z_1) - \omega_\phi(R, z_2)] = 2\textstyle\int_{\Sigma^2} \star_3 \Im\mathrm{m}\mathcal{W} &= \textstyle\int_{\Sigma^3} d\star_3 \Im\mathrm{m}\mathcal{W} \\ &= 2\textstyle\int_{\Sigma^3} d^3x \Im\mathrm{m}\,(\partial_{\underline{m}}\mathcal{W}_{\underline{m}})\,, \end{aligned} \tag{4.76}$$

where $\partial\Sigma^3 = \Sigma^2$. Thus, $\Im\mathrm{m}\,(\partial_{\underline{m}}\mathcal{W}_{\underline{m}}) \neq 0$ implies that ω_ϕ is singular on the string somewhere between z_1 and z_2. These singularities are related to the presence of NUT sources, since we can define the NUT charge contained in Σ^3 as the integral of $d\omega$ over $\Sigma^2 = \partial\Sigma^3$:

$$-8\pi N_\Sigma = \int_{\Sigma^2} d\omega = \int_{\Sigma^3} d^2\omega = 2\int_{\Sigma^3} d^3x \Im\mathrm{m}\,(\partial_{\underline{m}}\mathcal{W}_{\underline{m}})\,. \tag{4.77}$$

Thus, the condition $\Im\mathrm{m}\,(\partial_{\underline{m}}\mathcal{W}_{\underline{m}}) = 0$, required by supersymmetry, is equivalent to the absence of sources of NUT charge.

Hartle and Hawking argued that the only solutions in the IWP class with no NUT charge (and no singularities) were the Majumdar-Papapetrou solutions [106,107] which are regular and static. We will review their arguments in Section 4.3.1 and show that there are indeed non-trivial solutions that satisfy the KSIs and have no NUT charges, apart from the Majumdar-Papapetrou ones; they all have negative total mass, which causes other naked singularities to appear.

Thus, if we include positivity of all masses among the requirements necessary to have supersymmetry, the only supersymmetric black-holes-type solutions of pure $N = 2, d = 4$ supergravity will indeed be the Majumdar-Papapetrou solutions. We will have to consider more general $N = 2, d = 4$ theories in order to be able to have stationary solutions such as the one found in Ref. [45], that satisfy the KSIs and have positive mass. This will be done in Section 4.3.2.

Next, let us consider the KSI Eq. (4.17) which relates the sources of the scalar fields with those of the vector fields. If we consider only point-like sources and call Σ_A the scalar charge at $\vec{x}_A$, this equation implies, at each sources

$$\Sigma_A = 2e^{-i\alpha}\,\mathfrak{D}_i\mathcal{Z}|_{\vec{x}_A}\,. \tag{4.78}$$

As mentioned before, the scalar sources are completely determined by the electric and magnetic charges and the asymptotic values of the scalar fields. This is known as *secondary scalar hair* [100]. Primary scalar hair correspond to completely free parameters as in the Einstein-scalar solutions of Ref. [108] or in the solutions of Ref. [109] which may be embedded in $N = 4, d = 4$ supergravity. Neither of these solutions is supersymmetric (nor regular) and the above KSI explains just why.

But there is more to the above KSI: it shows that the existence of attractors at the sources implies total absence of scalar sources, either of primary or secondary type. Since this seems to be necessary in order to have regular event horizons, this KSI implies that there will not be supersymmetric black holes with scalar hair in these theories. Unfortunately, it seems possible to have singular supersymmetric solutions with primary scalar hair.

We can summarize the results obtained in this section as follows: we have identified a series of requirements necessary to avoid singularities in supersymmetric black-hole-type solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets, which can be associated to having unbroken supersymmetry everywhere (including the sources).

I The conditions

$$\Im\mathrm{m}\,(\vec{\nabla} \times \vec{\mathcal{W}}) = 0\,, \tag{4.79}$$

$$\Im\mathrm{m}\,(\vec{\nabla} \cdot \vec{\mathcal{W}}) = 0\,, \tag{4.80}$$

have to be satisfied everywhere in order to have supersymmetry everywhere. They ensure the absence of string singularities associated to source of NUT charge and other singularities associated to sources of angular momentum We stress that, when dealing with solutions, all finite contributions to the first equation should be ignored and the second equation can only have singular terms in the l.h.s.

II The mass has to be positive. Actually, the masses of each of the sources of the solutions should be positive. They cannot be rigorously defined in general (for multi-black-hole solutions), but they can be identified with certain confidence in the supersymmetric configurations at hands [110].

III The attractor equations (4.35) must be satisfied at each of the sources for admissible values of the scalars and the value of the central charge at each of them must be finite. As we have seen, the first condition is equivalent to the total absence of scalar sources.

The last two conditions are associated to the finiteness and positivity of $-g_{rr}$ outside the sources. Since $-g_{rr} \sim e^{-\mathcal{K}}$, it would be finite and positive as long as the scalar fields take admissible values within their domain of definition. All the zeroes of $-g_{rr}$ can be related to singularities of the scalar fields. Imposing that the scalar fields take admissible values everywhere is too strong a condition, since it is almost equivalent to directly impose absence of singularities in the metric.

The conditions that we have imposed are, however, heuristically equivalent: for a single black-hole solution the conditions of asymptotic flatness and positivity of the masses ensure positivity of $-g_{rr}$ in the limit $r \to \infty$. The third condition ensures positivity in the $r \to 0$ limit and, furthermore, ensures that there will be a horizon of finite area. Since there are no reasons to expect singularities at finite values of r, the positivity and finiteness should hold for all finite values of r. The same should happen in multi-black-hole solutions.

4.3 $N = 2, d = 4$ attractors, KSIs and BPS black-hole sources

Now we want to apply the results of the previous sections to several examples of black-hole-type solutions of $N = 2, d = 4$ supergravity theories, demanding the three conditions formulated in the introduction and checking the regularity of those solutions that satisfy them. We are going to start with the simplest theory.

4.3.1 Pure $N = 2, d = 4$ supergravity

This theory has $\bar{n} = 1$, no scalar fields, and it is given by the prepotential

$$\mathcal{F} = -\tfrac{i}{2}(\mathcal{X}^0)^2\,, \quad \Rightarrow \mathcal{F}_0 = -i\mathcal{X}^0\,. \tag{4.81}$$

This implies that the components of the symplectic section $\mathcal{V}$ are constant

$$\mathcal{L}^0 = i\mathcal{M}_0 = e^{i\gamma}/\sqrt{2}\,, \tag{4.82}$$

and X is not related to any Kähler potential, but

$$X = \frac{e^{i\gamma}}{\sqrt{2}}(\mathcal{L}^0/X)^{-1} = \frac{e^{i\gamma}}{\sqrt{2}(\mathcal{R}^0 + i\mathcal{I}^0)}\,. \tag{4.83}$$

The central charge is constant and given by

$$\mathcal{Z} = -\frac{ie^{i\gamma}}{\sqrt{2}}(p^0 - iq_0) \equiv -\frac{ie^{i\gamma}}{\sqrt{2}}\tilde{q}\,. \tag{4.84}$$

The attractor equations do not make sense because $\mathcal{Z}$ is already moduli-independent.

The timelike supersymmetric configurations of this theory were first found by Tod in his pioneering paper Ref. [11], belong to the family of solutions found by Perjés, Israel and Wilson (IWP) [14, 111]; they are completely determined by the choice of a single complex, harmonic function that we denote by $\tilde{\mathcal{I}}$. In the framework of general $N = 2, d = 4$ theories, the solutions of pure $N = 2, d = 4$ supergravity are given by just two real harmonic functions $\mathcal{I}^0$ and $\mathcal{I}_0$, the components of the real symplectic vector $\mathcal{I}$. The relation between $\mathcal{I}$ and $\tilde{\mathcal{I}}$ is

$$\tilde{\mathcal{I}} = \mathcal{I}^0 - i\mathcal{I}_0\,. \tag{4.85}$$

Observe that

$$X = -\frac{ie^{i\gamma}}{\sqrt{2}\tilde{\mathcal{I}}}\,, \tag{4.86}$$

and therefore $\sqrt{2}X$ coincides with the function V of Ref. [11] and is the inverse of the complex harmonic function.

It is convenient to use the complex formulation of this theory. In it, the symplectic product of two real symplectic vectors x, y can be written in the form $\langle\, x \mid y\,\rangle = \Im\mathrm{m}\,(\tilde{x}^*\tilde{y})$ where the tilde indicates complexification ($\tilde{x} = x^0 - ix_0$ etc.). Further, electric-magnetic duality rotations of the symplectic vectors is equivalent to multiplication by a global phase $\tilde{x}' = e^{i\gamma}\tilde{x}$. We would like to stress that the metric is invariant under these transformations.

Using Eq. (4.81) one finds that $\mathcal{R}$, the real part of $\mathcal{V}/X$ is the symplectic vector

$$\mathcal{R} = \left(\begin{array}{c} -\mathcal{I}_0 \\ \mathcal{I}^0 \end{array}\right), \;\Rightarrow \tilde{\mathcal{R}} = -i\tilde{\mathcal{I}}\,, \;\Rightarrow -g_{rr} = \frac{1}{2|X|^2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = |\tilde{\mathcal{I}}|^2\,. \tag{4.87}$$

Finally,

$$\vec{\mathcal{W}} = \tilde{\mathcal{I}}^*\vec{\nabla}\tilde{\mathcal{I}}\,. \tag{4.88}$$

It was argued by Hartle and Hawking [104] that the only regular black hole solutions in the IWP family are the static Majumdar-Papapetrou solutions that describe several charged black holes in static equilibrium. We are going to see that these are in fact the only solutions which are everywhere supersymmetric (condition I) and that demanding positivity of the masses of the components (condition II) is enough to have regular black holes (condition III plays no rôle here).

Single, static black hole solutions

The complex harmonic function $\tilde{\mathcal{I}}$ adequate to describe a static, spherically symmetric, extreme black hole with magnetic and electric charges p^0 and q_0 is

$$\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\infty + \frac{\tilde{q}}{r}\,, \qquad \tilde{q} \equiv p^0 - iq_0\,, \tag{4.89}$$

and asymptotic flatness requires $|\tilde{\mathcal{I}}_\infty| = 1$. Since $\tilde{\mathcal{I}}_\infty$ is just a phase that can be taken to be unity by an electric-magnetic duality rotation. Then,

$$-g_{rr} = |\tilde{\mathcal{I}}|^2 = 1 + \frac{2\Re\mathrm{e}(\tilde{\mathcal{I}}^*_\infty\tilde{q})}{r} + \frac{|\tilde{q}|^2}{r^2}\,. \tag{4.90}$$

The mass is given by

$$M = \Re\mathrm{e}(\tilde{\mathcal{I}}^*_\infty\tilde{q}) = \langle\, \mathcal{R}_\infty \mid q\,\rangle\,, \tag{4.91}$$

and the equations of motion and supersymmetry seem to allow for it to be positive or negative. When M is negative $|\tilde{I}|^2$ will vanish for some finite value of r, giving rise to a naked singularity. In the limit $r \to 0$, which makes sense if M is positive, we find that the area of the 2-spheres of constant t and r is finite and equal to

$$A = 4\pi|\tilde{q}|^2 = 8\pi|\mathcal{Z}|^2 . \tag{4.92}$$

Observe that, in general,

$$|M| \neq \sqrt{2}|\mathcal{Z}| , \tag{4.93}$$

even though these solutions are usually understood to be supersymmetric.

For this solution Eq. (4.79) is automatically satisfied, while Eq. (4.80) takes the form

$$\Im\mathrm{m}\,(\vec{\nabla}\cdot\vec{\mathcal{W}}) = -4\pi\Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q})\,\delta^{(3)}(\vec{x}) = 0 . \tag{4.94}$$

We can, either

1. Adopt the point of view proposed in this paper that the integrability condition has to be satisfied everywhere (condition I), whence impose the condition

$$\Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q}) = \langle\,\mathcal{I}_\infty \mid q\,\rangle = 0 . \tag{4.95}$$

$\tilde{\mathcal{I}}_\infty$ is just a phase and this condition determines it: $\tilde{\mathcal{I}}_\infty = \pm\tilde{q}/|\tilde{q}| \equiv e^{i\beta}$. The complex harmonic function becomes

$$\tilde{\mathcal{I}} = e^{i\beta}\left(1 \pm \frac{|\tilde{q}|}{r}\right) , \tag{4.96}$$

The overall phase $e^{i\beta}$ is irrelevant for our problem (it can always be eliminated by an electric-magnetic duality rotation that does not change the metric), but the relative sign between the two terms, which is the sign of the mass,

$$M = \pm|\tilde{q}| = \pm|\mathcal{Z}| , \tag{4.97}$$

is important since the minus sign leads to naked singularities. We take the positive sign as to comply with condition II. We can the integrate the equation for ω everywhere. The above condition, however, implies the vanishing of the r.h.s. of the equation and, therefore, also that of ω. Thus, after imposing conditions I and II we obtain a solution which is static and spherically symmetric and has a regular horizon if $M > 0$; Or

2. We can accept this singularity, ignoring condition I, arguing that, after all, the harmonic functions are already singular at that point[10] and proceed to integrate the equation and obtain ω which, in spherical coordinates, takes the form

$$\omega = 2N\cos\theta d\phi\,, \tag{4.98}$$

where N is NUT charge and it is given by

$$N = \Im\mathrm{m}\,(\tilde{\mathcal{I}}^{*}_{\infty}\tilde{q}) = \langle\, \mathcal{I}_{\infty} \mid q\,\rangle\,,\ \Rightarrow |M+iN| = \sqrt{2}|\mathcal{Z}_{\infty}|\,. \tag{4.99}$$

The metric is no longer static, but stationary, and contains either wire singularities or closed timelike curves plus Taub-NUT asymptotics.

It is clear that by imposing conditions I and II, these pathologies are avoided. Furthermore, in the microscopic models of black holes constructed in the framework of String Theory there seem to be no configurations that give rise to macroscopic NUT charge (nor to negative masses). The agreement between spacetime supersymmetry and the microscopic String Theory models on this point, together with the elimination of pathologies is encouraging and we will see that it applies to more cases.

Single black hole solutions with a dipole term

Let us now consider harmonic functions adequate to describe rotating supersymmetric black holes. We can add angular momentum to the previous solution by adding a dipole term to its complex harmonic function which becomes:

$$\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_{\infty} + \frac{\tilde{q}}{r} + (\vec{\tilde{m}}\cdot\vec{\nabla})\frac{1}{r}\,, \tag{4.100}$$

where $\vec{m} = (\vec{m}^0, \vec{m}_0)$ is a symplectic vector of dipole magnetic and electric momenta. When they are parallel we can take them to have only z component and, then, in spherical coordinates

$$\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_{\infty} + \frac{\tilde{q}}{r} - \frac{\tilde{m}\cos\theta}{r^2}\,. \tag{4.101}$$

The corresponding ω (which exists except at the singularities of $\tilde{\mathcal{I}}$) is

$$\omega = \left[2N\cos\theta + 2J\frac{\sin^2\theta}{r^2} + \Im\mathrm{m}(\tilde{q}^{*}\tilde{m})\frac{\sin^2\theta}{r^3}\right]d\phi\,. \tag{4.102}$$

[10] We have seen that the solution can, nevertheless, be regular at that point, which is the event horizon.

N is the NUT charge and is given again by Eq. (4.99). The new features are J, the z component of the angular momentum, given by

$$J = \Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty \tilde{m}) = \langle\, \mathcal{I}_\infty \mid m, \rangle\,, \tag{4.103}$$

and $\Im\mathrm{m}(\tilde{q}^*\tilde{m})$ which does not have a conventional name but vanishes when $N = J = 0$.

Let us now analyze the KSIs Eqs. (4.79) and (4.80) (condition I). In the general case they take, respectively, the form

$$2\left[\Im\mathrm{m}\,(\tilde{q}^*\nabla_m)\vec{\nabla}\frac{1}{r}\right] \times \vec{\nabla}\frac{1}{r} - i\left(\nabla_{m^*}\vec{\nabla}\frac{1}{r}\right) \times \left(\nabla_m\vec{\nabla}\frac{1}{r}\right) = 0\,, \tag{4.104}$$

$$\begin{aligned}\Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q})\,\delta^{(3)}(\vec{x}) + \Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\nabla_m)\,\delta^{(3)}(\vec{x}) + \frac{1}{r}\Im\mathrm{m}\,(\tilde{q}^*\nabla_m)\,\delta^{(3)}(\vec{x}) +& \\ +\delta^{(3)}(\vec{x})\Im\mathrm{m}\,(\tilde{q}\nabla_{m^*})\frac{1}{r} + \Im\mathrm{m}\left\{\left(\nabla_{m^*}\frac{1}{r}\right)\left(\nabla_m\delta^{(3)}(\vec{x})\right)\right\} =& \ 0\,, \end{aligned}\tag{4.105}$$

and are satisfied if

$$N = \Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q}) = \langle\, \mathcal{I}_\infty \mid q\,\rangle = 0\,, \tag{4.106}$$

$$\vec{J} = \Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\vec{\tilde{m}}) = \langle\, \mathcal{I}_\infty \mid \vec{m}\,\rangle = 0\,, \tag{4.107}$$

$$\Im\mathrm{m}\,(\tilde{q}^*\vec{\tilde{m}}) = \langle\, q \mid \vec{m}\,\rangle = 0\,, \tag{4.108}$$

$$\Im\mathrm{m}\,(\tilde{m}^*_{[\underline{m}}\tilde{m}_{\underline{n}]}) = \langle\, m_{[\underline{m}} \mid m_{\underline{n}]}\,\rangle = 0\,, \tag{4.109}$$

where we have defined the differential operator $\nabla_m \equiv \vec{\tilde{m}}\cdot\vec{\nabla}$ and where we have taken into account Eq. (4.103) to identify the angular momentum.

The first condition is, again, the absence of sources of NUT charge. The second condition is the absence of sources of angular momentum. The third and fourth conditions are automatically satisfied in this theory if the first two are.

In this case, these conditions are not enough to eliminate all the singularities introduced by the dipole term since the above conditions do not cancel terms like $|\vec{\tilde{m}}\cdot\vec{\nabla}\frac{1}{r}|^2$ in the g_{rr} component of the metric and we no longer find a regular 2-sphere in the $r\to 0$ limit. However, we are going to argue that, although technically possible, dipole terms should not be allowed in $\mathcal{I}$ because their only possible origin is a distribution of point-like charges and it is the fundamental distribution of point-like charges that we have to consider in the above equations and not the field they produce at distances larger than its size. It is in these conditions that imposing supersymmetry everywhere is equivalent to cosmic censorship.

Indeed, from the point of view of the electromagnetic fields, the magnetic dipole momenta, for instance, can have two fundamental origins: dipole momenta in a distribution of magnetic monopoles or fundamental dipole momenta that can be seen as stationary electric currents. In standard electrodynamics the first possibility is experimentally excluded (see, e.g. Ref. [112]) but in $N = 2, d = 4$ supersymmetric configurations it is the only one allowed (see Eq. (4.72)).

The supersymmetric Kerr-Newman solution

Therefore we must only consider distributions of static point-like charges. We will do so in a moment, but there is an interesting example of rotating black-hole-type solution which must be considered before: it is given by the complex harmonic function

$$\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\infty + \frac{\tilde{q}}{\tilde{r}}\,, \qquad \tilde{r} \equiv \sqrt{x^2 + y^2 + (z - i\alpha)^2}\,, \tag{4.110}$$

which is known to lead to the ("ultra-extreme") supersymmetric Kerr-Newman solution with angular momentum around the z axis; as is known it has naked singularities, as all 4-dimensional supersymmetric rotating "black-holes" [13]. This is the prototype of solution for which supersymmetry does not act as a "cosmic censor" as proposed in [20]. Generalizations of this solution in some other $N = 2, d = 4$ theories have been constructed in Ref. [96].

The asymptotic expansion of $\tilde{I}$

$$\tilde{\mathcal{I}} \sim \tilde{\mathcal{I}}_\infty + \frac{\tilde{q}}{r} - \frac{i\alpha\tilde{q}z}{r^3} + \cdots\,, \tag{4.111}$$

corresponds to a charge distribution with only two independent parameters: α and $\tilde{q}$. The magnetic (electric) dipole momentum is equal to the product of α and the electric (magnetic) charge and the infinite number of non-vanishing higher momenta depend also on these few parameters.

According to the point of view advocated here this solution should not be considered because it corresponds to the far field of a very charge distribution. As we are going to see, condition I is enough to exclude it.

Finding the sources of the solution associated to the above complex harmonic function is very complicated. To start with, $\tilde{\mathcal{I}}$ is singular on the ring $x^2 + y^2 = \alpha^2\,,\ z = 0$ but it is also discontinuous on a disk bounded by the ring (see e.g. [113], whose results we are going to use here. See also Refs. [114, 115].).

Eqs. (4.79) and (4.80), which express condition I, take, respectively, the form

$$\Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q})\Re\mathrm{e}\,(\vec{\nabla}\times\vec{C}) + \Re\mathrm{e}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q})\Im\mathrm{m}\,(\vec{\nabla}\times\vec{C}) + |\tilde{q}|^2\Im\mathrm{m}\left(\frac{1}{\tilde{r}^*}\vec{\nabla}\times\vec{C}\right) = 0\,, \tag{4.112}$$

$$\Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q})\Re\mathrm{e}\,(\vec{\nabla}\cdot\vec{C}) + \Re\mathrm{e}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q})\Im\mathrm{m}\,(\vec{\nabla}\cdot\vec{C}) + |\tilde{q}|^2\Im\mathrm{m}\left(\frac{1}{\tilde{r}^*}\vec{\nabla}\cdot\vec{C}\right) = 0\,, \tag{4.113}$$

where we have defined

$$\vec{C} \equiv \frac{(x, y, z - i\alpha)}{[x^2 + y^2 + (z - i\alpha)^2]^{3/2}}\,. \tag{4.114}$$

The curl and divergence of $\vec{C}$ have been carefully computed in Ref. [113] in a distributional sense, i.e. as integrals of their products with test functions. For us it is enough to known that

$$\Re\mathrm{e}\,(\vec{\nabla}\times\vec{C}) = \Im\mathrm{m}\,(\vec{\nabla}\cdot\vec{C}) = 0\,, \tag{4.115}$$

and that $\Im\mathrm{m}\,(\vec{\nabla}\times\vec{C})$ vanishes for vanishing α. We are left with

$$\left[\Re\mathrm{e}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q}) + |\tilde{q}|^2\Re\mathrm{e}\,\frac{1}{\tilde{r}}\right]\Im\mathrm{m}\,(\vec{\nabla}\times\vec{C}) = 0\,, \tag{4.116}$$

$$\left[\Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q}) - |\tilde{q}|^2\Im\mathrm{m}\,\frac{1}{\tilde{r}}\right]\Re\mathrm{e}\,(\vec{\nabla}\cdot\vec{C}) = 0\,. \tag{4.117}$$

The only way to satisfy the first condition is to have $\Im\mathrm{m}\,(\vec{\nabla}\times\vec{C}) = 0$, which requires $\alpha = 0$ (no sources of angular momentum). Since $\Re\mathrm{e}\,(\vec{\nabla}\cdot\vec{C}) \neq 0$ always, the only way to satisfy the second condition is to have $\Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty\tilde{q}) = 0$ as before (no sources of NUT charge) and $\Im\mathrm{m}\,\frac{1}{\tilde{r}} = 0$ which also requires $\alpha = 0$.

Thus, imposing supersymmetry everywhere is equivalent, yet again, to requiring absence of sources of NUT charge and angular momentum. In the supersymmetric Kerr-Newman solution all the angular momentum originates in that source[11] and, thus, that solution and its naked singularities can be excluded from the class of everywhere supersymmetric solutions of $N = 2, d = 4$ supergravity. Again, supersymmetry acts as a cosmic censor and, most importantly, there is agreement between the macroscopic description of black holes provided by Supergravity and the microscopic models

[11] We are going to see that there are solutions with angular momentum and no elementary sources of angular momentum.

provided by String Theory in which there seems to be no way of having angular momentum without breaking supersymmetry.

Therefore, we must only consider distributions of point-like charges, which correspond to complex harmonic functions of the form

$$\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\infty + \sum_A \frac{\tilde{q}_A}{|\vec{x} - \vec{x}_A|}\,, \tag{4.118}$$

from which dipole (and higher) momenta arise only in asymptotic expansions:

$$\tilde{\mathcal{I}} \sim \tilde{\mathcal{I}}_\infty + \frac{\sum_A \tilde{q}_A}{|\vec{x}|} + \frac{(\sum_A \tilde{q}_A \vec{x}_A)\cdot\vec{x}}{|\vec{x}|^3} + \cdots\,, \tag{4.119}$$

and may give rise to non-vanishing angular momentum

$$\vec{J} = \Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty \vec{\tilde{m}}) = \langle\, \mathcal{I}_\infty \mid \vec{m}\,\rangle\,, \qquad \vec{m} = -\sum_A q_A \vec{x}_A\,, \tag{4.120}$$

but not to non-vanishing NUT charge.

$$N = \Im\mathrm{m}\,(\tilde{\mathcal{I}}^*_\infty \tilde{q}) = \langle\, \mathcal{I}_\infty \mid q\,\rangle = 0\,, \qquad q = \sum_A q_A\,. \tag{4.121}$$

We are going to look for this kind of solutions in pure $N = 2, d = 4$ supergravity next, recovering the (negative) Hartle and Hawking result [104]. We will have to look for them in more general $N = 2, d = 4$ theories.

Solutions with two black holes

Let us consider, to start with, just two poles

$$\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\infty + \frac{\tilde{q}_1}{|\vec{x} - \vec{x}_1|} + \frac{\tilde{q}_2}{|\vec{x} - \vec{x}_2|}\,. \tag{4.122}$$

Asymptotic flatness requires $|\tilde{\mathcal{I}}_\infty| = 1$. The condition Eq. (4.79) is automatically satisfied and (4.80) takes the form

$$\left[\langle\, \mathcal{I}_\infty \mid q_1\,\rangle + \frac{\langle\, q_2 \mid q_1\,\rangle}{|\vec{x}_1 - \vec{x}_2|}\right]\delta^{(3)}(\vec{x} - \vec{x}_1) + \left[\langle\, \mathcal{I}_\infty \mid q_2\,\rangle + \frac{\langle\, q_1 \mid q_2\,\rangle}{|\vec{x}_2 - \vec{x}_1|}\right]\delta^{(3)}(\vec{x} - \vec{x}_2) = 0\,, \tag{4.123}$$

which leads to the two equations

$$
\begin{aligned}
\langle\, \mathcal{I}_\infty \mid q_1 \,\rangle + \frac{\langle\, q_2 \mid q_1 \,\rangle}{|\vec{x}_1 - \vec{x}_2|} &= 0\,, \\
\langle\, \mathcal{I}_\infty \mid q_2 \,\rangle + \frac{\langle\, q_1 \mid q_2 \,\rangle}{|\vec{x}_2 - \vec{x}_1|} &= 0\,,
\end{aligned}
\tag{4.124}
$$

each of which expresses the absence of sources of NUT charge at $\vec{x}_1$ and $\vec{x}_2$. The antisymmetry of the symplectic product implies the consistency condition

$$
\langle\, \mathcal{I}_\infty \mid q_1 + q_2 \,\rangle = 0\,, \tag{4.125}
$$

which means that the total charge of the two objects satisfies the same condition (no global NUT charge) as the charge of just one.

Expanding asymptotically $\mathcal{I}$ and using the above constraints we find that this two-body system has a total mass and angular momentum given by

$$
M = \sum_A \langle\, \mathcal{R}_\infty \mid q_A \,\rangle \equiv \sum_A M_A\,, \tag{4.126}
$$

$$
\vec{J} = \langle\, \mathcal{I}_\infty \mid \vec{m} \,\rangle = \langle\, q_1 \mid q_2 \,\rangle \frac{(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|}\,. \tag{4.127}
$$

Observe that there is total angular momentum even though there are no sources of angular momentum.

There are two types of solutions to these equations required by condition I:

1. Each object's charge satisfies the condition for single independent objects $\langle\, \mathcal{I}_\infty \mid q_A \,\rangle = 0$ which requires $\langle\, q_2 \mid q_1 \,\rangle = 0$. In this theory this means that the phases of $\tilde{\mathcal{I}}_\infty, \tilde{q}_1$ and $\tilde{q}_2$ are such that

   $$
   \tilde{I} = e^{i\beta}\left(1 + \sum_A \frac{s_A|\tilde{q}_A|}{|\vec{x} - \vec{x}_A|}\right)\,, \tag{4.128}
   $$

   where $s_A = \pm 1$. The total mass is given by the formula Eq. (4.91)

   $$
   M = \Re\mathrm{e}(\tilde{\mathcal{I}}^*_\infty \sum_A \tilde{q}_A) = \langle\, \mathcal{R}_\infty \mid \sum_A q_A \,\rangle = \sum_A s_A|\tilde{q}_A|\,, \tag{4.129}
   $$

   and the angular momentum vanishes (ω vanishes).

   These are the Majumdar-Papapetrou solutions [106, 107]. Only the solutions with all $s_A = +1$ are regular, but one could argue that only those correspond to objects that would have positive masses $M_A = |\tilde{q}_A|$ if they were isolated [110]. This is the meaning of condition II.

These solutions describe two charged, static black holes in equilibrium with their event horizons placed at $\vec{x}_1$ and $\vec{x}_2$ which are really 2-spheres of finite areas equal to $4\pi|\tilde{q}_1|^2$ and $4\pi|\tilde{q}_2|^2$. They are, as argued by Hartle and Hawking, and as we are going to see, the only regular black-hole-type solutions in the whole IWP family [104]

2. $\langle \mathcal{I}_\infty \mid q_A \rangle \neq 0$ and we have two objects that cannot exist independently in the vacuum $\mathcal{I}_\infty$ (i.e. we have a bound state). The distance between them is fixed by the condition of absence of sources of NUT charge to be

$$|\vec{x}_2 - \vec{x}_1| = \frac{\langle q_1 \mid q_2 \rangle}{\langle \mathcal{I}_\infty \mid q_1 \rangle}\,. \tag{4.130}$$

The sign of the r.h.s. can always be made positive by flipping the sign of $\mathcal{I}_\infty$, which is irrelevant for the moduli and for solving Eq. (4.125). Thus, this equation always has a solution. However, when all the above conditions have been satisfied, the total mass of the solution is negative. The simplest way to see this is by first making $\tilde{\mathcal{I}}_\infty = 1$ by a duality rotation that does not change the metric. After the duality rotation one finds $\tilde{q}'_A = M_A + iN_A$, meaning that they are complex combinations of the masses and NUT charges of each object. Using $N_2 = -N_1$, the above condition takes the form

$$N_1 + \frac{N_1 M_2 - N_2 M_1}{|\vec{x}_2 - \vec{x}_1|} = N_1\left(1 + \frac{M_1 + M_2}{|\vec{x}_2 - \vec{x}_1|}\right) = 0\,, \tag{4.131}$$

which has solution only for vanishing NUT charges or for negative total mass $M_1 + M_2$ which violates condition II and produces naked singularities. Thus, we cannot simultaneously satisfy conditions I and II for bound states with $\langle q_1 \mid q_2 \rangle \neq 0$.

This result can be generalized to solutions with more poles: let us consider first the 3-pole harmonic function

$$\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\infty + \frac{\tilde{q}_1}{|\vec{x} - \vec{x}_1|} + \frac{\tilde{q}_2}{|\vec{x} - \vec{x}_2|} + \frac{\tilde{q}_3}{|\vec{x} - \vec{x}_3|}\,. \tag{4.132}$$

The ω integrability condition leads to three equations (one to cancel the NUT charge at each pole) which can be written as a linear system for the N_As:

$$\begin{pmatrix} \left(1+\frac{M_2}{r_{12}}+\frac{M_3}{r_{14}}\right) & -\frac{M_1}{r_{12}} & -\frac{M_1}{r_{13}} \\ -\frac{M_2}{r_{12}} & \left(1+\frac{M_1}{r_{12}}+\frac{M_3}{r_{23}}\right) & -\frac{M_2}{r_{23}} \\ -\frac{M_3}{r_{13}} & -\frac{M_3}{r_{23}} & \left(1+\frac{M_1}{r_{13}}+\frac{M_2}{r_{23}}\right) \end{pmatrix} \begin{pmatrix} N_1 \\ \\ N_2 \\ \\ N_3 \end{pmatrix} = 0\,. \tag{4.133}$$

It is easy to see that the determinant of the matrix is +1 plus terms linear and quadratic in the masses, all with positive sign. It will never vanish if all the masses are positive. This argument can be easily generalized to a higher number of poles and, therefore we conclude that the only solutions satisfying conditions I and II are the Majumdar-Papapetrou solutions. This result should be read in a positive sense: no singular solutions are allowed by the conditions proposed in the introduction, even if only static solutions are allowed in this simple theory. To find solutions with angular momentum satisfying conditions I-III we need to consider theories with scalars.

4.3.2 General $N = 2, d = 4$ supergravity

The setup of our problem in general $N = 2, d = 4$ theories is similar to pure supergravity case. Let us first consider spherically-symmetric, static, single black-hole-type solutions with magnetic and electric charges p^Λ and q_Λ. They are determined by a symplectic vector of $2\bar{n}$ real harmonic functions

$$\mathcal{I} = \begin{pmatrix} \mathcal{I}^\Lambda \\ \mathcal{I}_\Lambda \end{pmatrix} = \mathcal{I}_\infty + \frac{q}{r}\,, \qquad q \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}\,, \qquad \mathcal{I}_\infty \equiv \begin{pmatrix} \mathcal{I}^\Lambda_\infty \\ \mathcal{I}_{\Lambda\,\infty} \end{pmatrix}\,. \tag{4.134}$$

We assume that the stabilization equations have been solved and $\mathcal{R}(\mathcal{I})$ has been found in order to be able to construct the fields of the solutions.

The n complex scalars are constructed using the general formula Eq. (4.5). The moduli (the values of the n complex scalars Z^i at infinity, Z^i_∞) are complicated functions $Z^i_\infty(\mathcal{I}_\infty)$ of these $2n+2$ real constant components of $\mathcal{I}_\infty$. One of the components of $\mathcal{I}_\infty$ can be determined as a function of the remaining $2n+1$ by imposing asymptotic flatness of the metric, that is, $\langle\, \mathcal{R}_\infty \mid \mathcal{I}_\infty \,\rangle = 1$, and another one can be determined by imposing condition I, since Eq. (4.73) implies

$$N = \langle\, \mathcal{I}_\infty \mid q \,\rangle = 0\,. \tag{4.135}$$

It should always be possible to give the $2n$ real moduli any admissible value within their definition domain with the remaining $2n$ unconstrained real components of $\mathcal{I}_\infty$.

This is difficult to prove explicitly due to the complicated and theory-dependent relations between $\mathcal{I}_\infty$ and the moduli Z^i_∞, but it is safe to assume that in general it is possible.

Let us turn to condition II. The positivity of the masses, which is given by the general expression Eq. (4.53) has to be imposed by hand and, although this can always be done, it is a non-trivial constraint on the charges and moduli. The positivity of the masses can be also understood as part of a stronger requirement that the scalar fields take values only within their definition domain for all values of r. Actually, this requirement should suffice to ensure the finiteness of $-g_{rr}$ for $r \neq 0$.

The finiteness of $-g_{rr}$ for $r \neq 0$ is not enough to have a black hole and condition III has to be imposed to find a finite horizon area at $r = 0$.

If we want to describe more than one black hole we have to use harmonic functions with two point-like singularities:

$$\mathcal{I} = \mathcal{I}_\infty + \frac{q_1}{|\vec{x} - \vec{x}_1|} + \frac{q_2}{|\vec{x} - \vec{x}_2|} \,. \tag{4.136}$$

Again, one of the components of $\mathcal{I}_\infty$ is determined by imposing asymptotic flatness. Condition I now leads to the two equations Eqs. (4.124) which should determine another component of $\mathcal{I}_\infty$ and the parameter $|\vec{x}_1 - \vec{x}_2|$ if $\langle q_2 \mid q_1 \rangle \neq 0$. The question now is whether these solutions can be obtained while maintaining the positivity of the masses (condition II)

$$M_i \equiv \langle\, \mathcal{R}_\infty \mid q_i \,\rangle > 0 \,, \tag{4.137}$$

and solving the attractor equations for each of the singularities of the harmonic functions. We have no general answer to these questions and, what we are going to do is to study how the three conditions can actually be imposed in a particularly simple example and suffice to ensure regularity of the solutions.

A toy model with a complex scalar field

We are going to consider the $\bar{n} = 2$ theory with prepotential

$$\mathcal{F} = -i\mathcal{X}^0\mathcal{X}^1 \,. \tag{4.138}$$

This theory has only one complex scalar

$$\tau \equiv i\mathcal{X}^1/\mathcal{X}^0 \,, \tag{4.139}$$

in terms of which the period matrix is given by

$$(\mathcal{N}_{\Lambda\Sigma}) = \begin{pmatrix} -\tau & 0 \\ 0 & 1/\tau \end{pmatrix} \tag{4.140}$$

and, in the $\mathcal{X}^0 = i/2$ gauge, the Kähler potential and metric are

$$\mathcal{K} = -\ln \Im\mathrm{m}\tau\,, \qquad \mathcal{G}_{\tau\tau^*} = (2\Im\mathrm{m}\tau)^{-2}\,. \tag{4.141}$$

The reality of the Kähler potential requires the positivity of $\Im\mathrm{m}\tau$. Therefore, τ parametrizes the coset $SL(2,\mathbb{R})/SO(2)$ and can be identified with the *axidilaton* and this theory is a truncation of the $SO(4)$ formulation of $N=4, d=4$ supergravity.

The symplectic section $\mathcal{V}$ is

$$\mathcal{V} = \frac{1}{2(\Im\mathrm{m}\tau)^{1/2}} \begin{pmatrix} i \\ \tau \\ -i\tau \\ 1 \end{pmatrix}, \tag{4.142}$$

and the central charge is

$$\mathcal{Z}(\tau,\tau^*,q) = \langle \mathcal{V} \mid q \rangle = \frac{1}{2(\Im\mathrm{m}\tau)^{1/2}}[(p^1 - iq_0) - (q_1 + ip^0)\tau]\,. \tag{4.143}$$

The attractor equation is

$$\frac{d}{d\tau}\frac{1}{\Im\mathrm{m}\tau}[(p^1 - iq_0) - (q_1 + ip^0)\tau]\Big|_{\tau=\tau_{\text{fix}}} = 0\,, \tag{4.144}$$

and has the general solution

$$\tau_{\text{fix}} = \frac{p^1 + iq_0}{q_1 - ip^0}\,, \tag{4.145}$$

which is admissible (belongs to the definition domain of τ) if

$$\Im\mathrm{m}\tau_{\text{fix}} = p^0p^1 + q_0q_1 > 0\,. \tag{4.146}$$

The central charge at the fixed point of the scalar takes the value

$$\mathcal{Z}_{\text{fix}} = -i\frac{q_1 + ip^0}{|q_1 + ip^0|}\sqrt{p^0p^1 + q_0q_1}\,, \tag{4.147}$$

and it is always finite for $\tau_{\text{fix}} \neq 0$.

<u>*Solutions with a single black hole*</u>

Let us now consider solutions with

$$\mathcal{I} = \mathcal{I}_\infty + \frac{q}{r}\,. \tag{4.148}$$

In this theory the stabilization equations can be easily solved and they lead to

$$\mathcal{R} = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \mathcal{I}\,, \quad \Rightarrow \quad -g_{rr} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = 2(\mathcal{I}^0\mathcal{I}^1 + \mathcal{I}_0\mathcal{I}_1)\,, \tag{4.149}$$

which shows that the area of the horizon (if any) is related to $|\mathcal{Z}_{\text{fix}}|^2$ above according to the general formula Eq. (4.55).

We also have

$$\tau = i\frac{\mathcal{L}^1/X}{\mathcal{L}^0/X} = \frac{\mathcal{I}^1 + i\mathcal{I}_0}{\mathcal{I}_1 - i\mathcal{I}^0}\,, \tag{4.150}$$

which implies that the 4 harmonic functions are not entirely independent but have to satisfy

$$\Im\text{m}\tau = \mathcal{I}^0\mathcal{I}^1 + \mathcal{I}_0\mathcal{I}_1 > 0\,, \tag{4.151}$$

which ensures that, if there are no pathologies that make a black-hole interpretation of the solution impossible, the attractor equations will always have solutions and $\mathcal{Z}_{\text{fix}} \neq 0$. Thus, we will not have to worry about condition III but only about the positive definiteness of $\Im\text{m}\tau$.

The only possible pathologies (negative mass and presence of NUT charge) are clearly avoided by imposing conditions I and II, which is always possible and presents no difficulties.

Solutions with two black holes

Let us now consider solutions of the form

$$\mathcal{I} = \mathcal{I}_\infty + \frac{q_1}{r_1} + \frac{q_2}{r_2}\,, \qquad r_i \equiv |\vec{x} - \vec{x}_i|\,. \tag{4.152}$$

Our goal is to find a configuration (i.e. a set of asymptotic values $\mathcal{I}_\infty$ and charges $q_{1,2}$) that satisfy conditions I-III. The previous discussions indicate how this has to be done and which formulas need to be applied. There is no systematic procedure to find such a configuration but it is not too difficult to find one:

$$
\begin{aligned}
\mathcal{I}^0 &= \frac{1}{\sqrt{2}} + \frac{q}{r_1} + \frac{q}{r_2}\,, \\
\mathcal{I}^1 &= \frac{1}{\sqrt{2}} + \frac{8q}{r_1} + \frac{8q}{r_2}\,, \\
\mathcal{I}_0 &= -\frac{4q}{r_2}\,, \\
\mathcal{I}_1 &= -\frac{1}{4\sqrt{2}} - \frac{q}{r_1} + \frac{q}{r_2}\,,
\end{aligned}
\tag{4.153}
$$

where $q > 0$ in order to guarantee Eq. (4.151). The metric component

$$
-g_{rr} = 1 + \frac{9\sqrt{2}q}{r_1} + \frac{10\sqrt{2}q}{r_2} + \frac{16q^2}{r_1^2} + \frac{8q^2}{r_2^2} + \frac{40q^2}{r_1 r_2}\,,
\tag{4.154}
$$

is finite everywhere outside $r_{1,2} = 0$, and therefore, so is $\Im\mathrm{m}\tau$. In particular the "mass" of each of the two objects is positive

$$
M_1 = 9q/\sqrt{2}\,, \qquad M_2 = 5\sqrt{2}q\,, \qquad M = M_1 + M_2 = 19q/\sqrt{2}\,,
\tag{4.155}
$$

and in the $r_{1,2} \to 0$ limits we find spheres of finite areas

$$
\frac{A_1}{4\pi} = 16q^2 = 2|\mathcal{Z}_{\text{fix},1}|^2\,, \qquad \frac{A_2}{4\pi} = 8q^2 = 2|\mathcal{Z}_{\text{fix},2}|^2\,.
\tag{4.156}
$$

The total horizon area is

$$
\frac{A}{4\pi} = \frac{A_1}{4\pi} + \frac{A_2}{4\pi} = 24q^2 < 2|\mathcal{Z}_{\text{fix,tot}}|^2 = 64q^2\,,
\tag{4.157}
$$

which is the area of the horizon of a single black hole having the sum of the charges of the two black holes.

For this configuration

$$
\langle\, \mathcal{I}_\infty \mid q_1 \,\rangle = -\langle\, \mathcal{I}_\infty \mid q_2 \,\rangle = -q/\sqrt{2}\,, \qquad \langle\, q_2 \mid q_1 \,\rangle = 12q^2\,,
\tag{4.158}
$$

so, choosing

$$
r_{12} = |\vec{x}_2 - \vec{x}_1| = 12\sqrt{2}q\,,
\tag{4.159}
$$

we satisfy condition I (no NUT charges). The system has nevertheless angular momentum given by the general formula Eq. (4.127):

$$|J| = |\langle\, q_2 \mid q_1 \,\rangle| = 12q^2\,. \tag{4.160}$$

5

Discussion and conclusions

We have succeed in the characterization of supersymmetric solutions of general matter-coupled $N = 1$, $d = 5$ Supergravity and pure $N = 4$, $d = 4$ Supergravity. To this end, we have used the method of spinor bilinears. As usual, this method leads to separate the solutions between the time-like and null case. In the time-like case there are typically massive solitons whereas in the null case there are pp-waves.

$N = 1$, $d = 5$ Supergravity had been studied before by several authors. We have presented the first complete analysis with hyperscalars (the study of the theory with hyperscalars was initiated in Ref. [1, 2]). In the time-like case, the main novelty of the presence of hyperscalars is the enhancing of the holonomy of the spatial base manifold from $SU(2)$ to the full $SO(4)$ group, being the anti-self dual component of the spin connection related to the other fields. Indeed, in the ungauged case it is just the pull-back of the $\mathfrak{su}(2)$ connection of the quaternionic Kähler manifold (the same relation holds in the gauged case, but with some corrections). The condition on the hyperscalars to have unbroken supersymmetry has a very simple and suggestive form, indeed in the ungauged case it is the equation for quaternionic maps between hyperKähler manifolds (although the base manifold is not necessarily hyperKähler). Due to its simplicity, this equation could be the starting point to construct new, concrete supersymmetric solutions of this theory.

There were not previous analysis on the characterization of supersymmetric solutions of matter-coupled $N = 1$, $d = 5$ Supergravity belonging to the null case. We have found that in this case the spin connection of the three-dimensional subspace transverse to the wave is also related to the other fields. In the ungauged case it is again the pullback of the $\mathfrak{su}(2)$ connection of the quaternionic Kähler manifold. Equally, the condition on the hyperscalars is quite simple.

We found, in a very precise way, the generic projections to be imposed on the Killing spinors in order to have unbroken supersymmetry. In the time-like case of this theory all supersymmetric configurations preserve at least $1/8$ of the supersymmetries.

We have found solutions with one additional isometry in the time-like case which are the generalization of the Gibbons-Hawking instanton metric. As we mention, the presence of hyperscalars destroys the self-duality of the connection, this fact is reflected on the non-triviality of the three-dimensional connection, unlike the Gibbons-Hawking instanton which has flat three-dimensional metric.

It would be interesting to study the attractor mechanism and the entropy of the black hole solutions in presence of hyperscalars. Moreover, the pp-wave of the null class solutions can be dimensionally reduced to supersymmetric $N = 2$, $d = 4$ black holes. This raises new questions about how the 4-dimensional attractor mechanism is implemented in the 5-dimensional setting, taking into account that these 5-dimensional solutions belong to the null class and the standard attractor mechanism is proved only for solutions in the time-like class. The 5-dimensional origin of the 4-dimensional entropy can (and must) be investigated.

Moreover, the dimensional reduction of all the five-dimensional supersymmetric solutions to four dimensions can be performed. It would be interesting to see how this can be achieved in the framework of the characterization of supersymmetric solutions (such a characterization for the $N = 2$, $d = 4$ theory coupled to matter has been done in Refs. [3, 4]). In addition, the reduction/uplifting of supersymmetric solutions can be analyzed together with the six-dimensional theory. Therefore theories with $N = 2$ supersymmetries in six, five and four dimensions can be treated in an unified frame.

In the $N = 4$, $d = 4$ theory we have determined (in the time-like case) the precise way in which the three classes of holonomy of the base manifold (flat, $U(1)$ and $SU(2)$) arise, i. e. we have indicated how the spin connection is related to the other variables. We have thus extended the work of Tod [5] who characterized only solutions with flat holonomy in the base space. Here, the symmetries of the theory ($SU(4)$ and $SL(2, \mathbb{R})$) play a crucial role, guiding us in the construction of the supersymmetric configurations and solutions.

The formalism we have developed to analyze the $N = 4$, $d = 4$ theory can be adapted to other four dimensional theories with more supersymmetries, that is $N = 6$ and 8 four-dimensional supergravities.

Another interesting extension of our work would be to perform the characterization with R^2 corrections both in four and five dimensions. This is particularly viable because the supersymmetry variations are the same when R^2 corrections are considered (although the equation of motion changes).

We have seen that the general Killing Spinor Identities (KSIs) found in Ref. [6] can be used to obtain useful relations between the equations of motion evaluated on supersymmetric configurations. This is a very powerful tool, it has allowed us for example to avoid the evaluation of (some components of) the Einstein equations. Moreover, the KSIs can be computed in any theory of supergravity. Other authors have used analogous relations between equations of motion, but they have found them by using directly the integrability conditions of the Killing spinor equations, which is a way harder than the KSIs.

We have also shown how the supersymmetry acts as cosmic censor. By demanding that supersymmetry is unbroken everywhere, even at the sources, the configurations are constrained in such a way that many pathological solutions (naked singularities) can be discarded. We have formulated the condition of having unbroken supersymmetry everywhere by means of three conditions that supersymmetric black-hole-type solutions have to satisfy. We have shown how these conditions constrain the possible sources by, basically, excluding those with NUT charge, angular momentum, negative energy and scalar hair, which seemingly cannot be modeled in String Theory. We arrived at a picture in which if an observer far away from one of the globally supersymmetric configurations we have considered, detects angular momentum and non-trivial scalar fields he/she will only find static electromagnetic sources in equilibrium when approaching the system.

These conditions and this picture should be improved by considering quantum corrections. Another interesting course of action would be to consider regularity of black-hole solutions in $N > 2$ theories, e.g. [7–9], and investigate the role played by the attractor [10].

A

Conventions and some formulae

A.1 General conventions

A.1.1 Notation

- We use the mostly minus signature $(+-\ldots-)$. η_{ab} is the Minkowski metric. Lowercase greek letters μ, ν, ρ... are space-time curved indices and lowercase latin letters a, b, c... are space-time flat (tangent) indices. When dealing with the spatial sector we use lowercase latin indices like i, j, k... which are underlined when are curved indices.

- Sometimes we indicate the contraction (without numerical weight) of tensors by a central dot, $A \cdot B = A^{\alpha\beta\gamma\cdots} B_{\alpha\beta\gamma\cdots}$.

- The internal product of a k-form and a vector is

$$i_V \omega \equiv \omega(V,\ ,\ldots,) \,, \qquad (i_V\omega)_{\mu_1\cdots\mu_{k-1}} = V^\alpha \omega_{\alpha\mu_1\cdots\mu_{k-1}} \tag{A.1}$$

- Symmetrization and anti-symmetrization is made with unit weight

$$(a_1 \cdots a_n) \equiv \frac{1}{n!} \sum_P P\,(a_1 \cdots a_n)\ , \tag{A.2}$$

$$[a_1 \cdots a_n] \equiv \frac{1}{n!} \sum_P \mathrm{sgn}(P) P\,(a_1 \cdots a_n)\ , \tag{A.3}$$

  where P is any permutation. For example $[ab] = \frac{1}{2}\,(ab - ba)$.

- k-forms are normalized according to

$$F = \frac{1}{k!} F_{\mu_1 \cdots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \ . \tag{A.4}$$

The value of the completely antisymmetric symbol is $\epsilon^{012\cdots} = \epsilon_{\underline{012\cdots}} = +1$.

The Hodge dual of a k-form is denoted by $\star F$ and it is given by

$$\star F^{\mu_1 \cdots \mu_{(d-k)}} = \frac{1}{k!\sqrt{|g|}} \epsilon^{\mu_1 \cdots \mu_d} F_{\mu_{(d-k+1)} \cdots \mu_d} \tag{A.5}$$

A.1.2 Affine and spin connection

∇ is the covariant derivative under general coordinate transformations and local Lorentz transformations, hence it is made from the affine connection $\Gamma_{\mu\nu}{}^{\alpha}$ and the spin connection $\omega_{\mu a}{}^{b}$. The covariant derivatives on tensors and spinors are

$$\nabla_\mu A^\nu = \partial_\nu A^\nu + \Gamma_{\mu\alpha}{}^{\nu} A^\alpha \tag{A.6}$$

$$\nabla_\mu A^a = \partial_\nu A^a + \omega_{\mu b}{}^{a} A^b \tag{A.7}$$

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \psi \ . \tag{A.8}$$

The curvature of the torsionless affine connection is defined by

$$[\nabla_\mu, \nabla_\nu] A^\alpha = R_{\mu\nu\beta}{}^{\alpha}(\Gamma) A^\beta \tag{A.9}$$

$$[\nabla_\mu, \nabla_\nu] \psi = -\frac{1}{4} R_{\mu\nu}{}^{ab}(\omega) \gamma_{ab} \psi \ . \tag{A.10}$$

Explicitely it yields

$$R_{\mu\nu\alpha}{}^{\beta}(\Gamma) = 2\partial_{[\mu} \Gamma_{\nu]\alpha}{}^{\beta} - 2\Gamma_{[\mu|\alpha}{}^{\rho} \Gamma_{\nu]\rho}{}^{\beta} \ , \tag{A.11}$$

$$R_{\mu\nu a}{}^{b}(\omega) = 2\partial_{[\mu} \omega_{\nu]a}{}^{b} - 2\omega_{[\mu|a}{}^{c} \omega_{\nu]c}{}^{b} \ . \tag{A.12}$$

The vielbein postulate

$$\nabla_\mu e_\nu{}^{a} = 0 \ , \tag{A.13}$$

relates the affine and spin connections

$$\omega_{\mu a}{}^{b} = \Gamma_{\mu\alpha}{}^{\beta} e_a{}^{\alpha} e_\beta{}^{b} + e_a{}^{\alpha} \partial_\mu e_\alpha{}^{b} \ . \tag{A.14}$$

In turn, the curvatures are related homogeneously

$$R_{\mu\nu\alpha}{}^{\beta}(\Gamma) = R_{\mu\nu a}{}^{b}(\omega) e_\alpha{}^{a} e_b{}^{\beta} \ . \tag{A.15}$$

Finally, metric compatibility and torsionlessness fully determine the connections to be of the form

$$\Gamma_{\mu\nu}{}^{\rho} = \tfrac{1}{2} g^{\rho\sigma} \left(2\partial_{(\mu} g_{\nu)\sigma} - \partial_\sigma g_{\mu\nu}\right) . \tag{A.16}$$

A.2 Special conventions in four dimensions

A.2.1 Complex (anti)-self-dual forms

For any 4-dimensional 2-form, we define

$$F^{\pm} \equiv \tfrac{1}{2}(F \pm i \star F) , \qquad F^{\pm} = \pm i \star F^{\pm} . \tag{A.17}$$

For any two 2-forms F, G, we have

$$F^{\pm} \cdot G^{\mp} = 0 , \qquad F^{\pm}{}_{[\mu}{}^{\rho} \cdot G^{\mp}{}_{\nu]\rho} = 0 . \tag{A.18}$$

A.2.2 Electric and magnetic components

Given any 2-form $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and a non-null 1-form $\hat{V} = V_\mu dx^\mu$, we can express F in the form

$$F = V^{-2}[E \wedge \hat{V} - \star(B \wedge \hat{V})] , \qquad E_\mu \equiv F_{\mu\nu} V^\nu , \qquad B_\mu \equiv \star F_{\mu\nu} V^\nu . \tag{A.19}$$

For the complex combinations $F^{\pm}$ we have

$$F^{\pm} = V^{-2}[C^{\pm} \wedge \hat{V} \pm i \star(C^{\pm} \wedge \hat{V})] , \qquad C^{\pm}_\mu \equiv F^{\pm}_{\mu\nu} V^\nu . \tag{A.20}$$

This decomposition is particularly useful in the time-like case where V generates the time translations, hence the above is a electric/magnetic decomposition of the Maxwell field in a covariant way.

It is interesting to study the compatibility of the above decomposition with the $SL(2,\mathbb{R})$ symmetry, which acts on F and $\star F$ in different ways. Let $\mathtt{M}$ be the matrix

$$\mathtt{M} \equiv \frac{1}{\Im\mathrm{m}\tau} \begin{pmatrix} |\tau|^2 & \Re\mathrm{e}\tau \\ \Re\mathrm{e}\tau & 1 \end{pmatrix} . \tag{A.21}$$

$SL(2,\mathbb{R})$ acts on $\mathtt{M}$ linearly

$$\mathtt{M}' = \Lambda \mathtt{M} \Lambda^T \tag{A.22}$$

where Λ is a $SL(2,\mathbb{R})$ matrix,

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{A.23}$$

If we solve $\{F, \star F\}$ in terms of $\{F, \tilde{F}\}$ then the decomposition (A.19) becomes explicitely $SL(2,\mathbb{R})$-covariant,

$$\begin{pmatrix} \tilde{F} \\ F \end{pmatrix} = \left[i_V \begin{pmatrix} \tilde{F} \\ F \end{pmatrix} \wedge V - \mathtt{M} S \star \left(i_V \begin{pmatrix} \tilde{F} \\ F \end{pmatrix} \wedge V \right) \right] \tag{A.24}$$

where S is the canonical antisymmetric matrix,

$$S \equiv \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \tag{A.25}$$

which is preserved by $SL(2,\mathbb{R})$ by definition,

$$\Lambda^T S \Lambda = S . \tag{A.26}$$

A.2.3 Null tetrads

If we have a (real) null vector l^μ, we can always add three more null vectors $n^\mu, m^\mu, \bar{m}^\mu$ to construct a complex null tetrad such that the local metric in this basis takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{A.27}$$

with the ordering $(l, n, m, \bar{m})$. For the local volume element we obtain $\epsilon^{lnm\bar{m}} = i$. The general expansion in the dual basis of 1-forms $\left(\hat{l}, \hat{n}, \hat{m}, \hat{\bar{m}}\right)$ of F^+ depends on three arbitrary complex functions a, b, c

$$F^+ = a\left(\hat{l} \wedge \hat{n} + \hat{m} \wedge \hat{\bar{m}}\right) + b\hat{l} \wedge \hat{\bar{m}} + c\hat{n} \wedge \hat{m} \,, \qquad F^- = (F^+)^* \,. \tag{A.28}$$

Then, in this case, F is not completely determined by its contraction with the null vector l, but

$$F^+ = L^\pm \wedge \hat{n} \pm {}^\star(L^\pm \wedge \hat{n}) + b\hat{l} \wedge \hat{m} \,, \qquad L^\pm_\mu \equiv F^\pm{}_{\mu\nu} l^\nu = a l_\mu - c m_\mu \,. \tag{A.29}$$

A.2.4 Raising and lowering $SU(4)$ indices

In $SU(4)$ two representations which are complex conjugates transform in inverse ways. Hence we may define an invariant scalar product in $SU(4)$ by complex conjugation. Complex conjugation lows and raises the $SU(4)$ indices, $X^I = {X_I}^*$. Thus, one immediately notice that a product like

$$A^I B_I$$

is invariant under $SU(4)$.

Besides the scalar product, there is a further $SU(4)$ invariant: the totally antisymmetric symbol ϵ_{IJKL}, $\epsilon_{1234} = +1$. It is real, hence is the same with upper and bottom indices. For a $SU(4)$ tensor we define the $SU(4)$ dual

$$\tilde{M}_{IJ} = \frac{1}{2}\epsilon_{IJKL}M^{KL} \ , \quad \tilde{M}^{IJ} = \frac{1}{2}\epsilon^{IJKL}M_{KL} \ . \tag{A.30}$$

B

Gamma matrices, bilinears and Fierz identities

B.1 Four dimensions

B.1.1 Gamma matrices and spinors

We work with a purely imaginary representation

$$\gamma^{a\,*} = -\gamma^a \,, \tag{B.1}$$

and our convention for their anticommutator is

$$\{\gamma^a, \gamma^b\} = +2\eta^{ab} \,. \tag{B.2}$$

Thus,

$$\gamma^0 \gamma^a \gamma^0 = \gamma^{a\,\dagger} = \gamma^{a\,-1} = \gamma_a \,. \tag{B.3}$$

The chirality matrix is defined by

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \tfrac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d \,, \tag{B.4}$$

and satisfies

$$\gamma_5{}^\dagger = -\gamma_5{}^* = \gamma_5 \,, \qquad (\gamma_5)^2 = 1 \,. \tag{B.5}$$

With this chirality matrix, we have the identity

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]} i}{(4-n)!} \epsilon^{a_1\cdots a_n b_1 \cdots b_{4-n}} \gamma_{b_1\cdots b_{4-n}} \gamma_5 \,. \tag{B.6}$$

Our convention for Dirac conjugation is

$$\bar{\psi} = i\psi^\dagger \gamma_0 \,. \tag{B.7}$$

Using the identity Eq. (B.6) the general $d = 4$ Fierz identity for *commuting* spinors takes the form

$$\begin{aligned}(\bar{\lambda} M \chi)(\bar{\psi} N \varphi) &= \tfrac{1}{4}(\bar{\lambda} M N \varphi)(\bar{\psi}\chi) + \tfrac{1}{4}(\bar{\lambda} M \gamma^a N \varphi)(\bar{\psi}\gamma_a \chi) - \tfrac{1}{8}(\bar{\lambda} M \gamma^{ab} N \varphi)(\bar{\psi}\gamma_{ab}\chi) \\ &\quad -\tfrac{1}{4}(\bar{\lambda} M \gamma^a \gamma_5 N \varphi)(\bar{\psi}\gamma_a\gamma_5\chi) + \tfrac{1}{4}(\bar{\lambda} M \gamma_5 N \varphi)(\bar{\psi}\gamma_5\chi)\,.\end{aligned} \tag{B.8}$$

We use 4-component chiral spinors whose chirality is related to the position of the $SU(4)$ index:

$$\gamma_5 \chi_I = +\chi_I \,, \qquad \gamma_5 \psi_{\mu\, I} = -\psi_{\mu\, I} \,, \qquad \gamma_5 \epsilon_I = -\epsilon_I \,. \tag{B.9}$$

Both (chirality and position of the $SU(4)$ index) are reversed under complex conjugation:

$$\gamma_5 \chi^*_I \equiv \gamma_5 \chi^I = -\chi^I \,, \qquad \gamma_5 \psi^*_{\mu\, I} \equiv \gamma_5 \psi_\mu{}^I = +\psi_\mu{}^I \,, \qquad \gamma_5 \epsilon^*_I \equiv \gamma_5 \epsilon^I = +\epsilon^I \,. \tag{B.10}$$

We take this fact into account when Dirac-conjugating chiral spinors:

$$\bar{\chi}^I \equiv i(\chi_I)^\dagger \gamma_0 \,, \quad \bar{\chi}^I \gamma_5 = -\bar{\chi}^I \,, \quad \text{etc.} \tag{B.11}$$

The sum of the two chiral spinors related by complex conjugation gives a standard (real) Majorana spinor with an $SU(4)$ index with the complicated transformation rule of Ref. [88].

Explicit gamma matrices according to our conventions are

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} , \\ \gamma^3 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \quad \gamma^5 = \begin{pmatrix} 0 & -i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}\end{aligned} \tag{B.12}$$

where σ^i are Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.13}$$

B.1.2 Fierz identities for bilinears

Here we are going to work with an arbitrary number N of chiral spinors, although we are ultimately interested in the $N = 4$ case only. Whenever there are special results for particular values of N, we will explicitly say so. We should bear in mind that the maximal number of independent chiral spinors is 2 and, for $N > 2$ (in particular for $N = 4$) N spinors cannot be linearly independent at a given point. This trivial fact has important consequences.

Given N chiral commuting spinors ϵ_I and their complex conjugates ϵ^I we can constructed the following bilinears that are not obviously related via Eq. (B.6):

1. A complex matrix of scalars

$$M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J\,, \qquad M^{IJ} \equiv \bar{\epsilon}^I \epsilon^J = (M_{IJ})^*\,, \tag{B.14}$$

   which is antisymmetric $M_{IJ} = -M_{JI}$.

2. A complex matrix of vectors

$$V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J\,, \qquad V_I{}^J{}_a \equiv i\bar{\epsilon}_I \gamma_a \epsilon^J = (V^I{}_{J\,a})^*\,, \tag{B.15}$$

   which is Hermitean:

$$(V^I{}_{J\,a})^* = V_I{}^J{}_a = V^J{}_{I\,a} = (V^I{}_{J\,a})^T\,. \tag{B.16}$$

3. A complex matrix of 2-forms

$$\Phi_{IJ\,ab} \equiv \bar{\epsilon}_I \gamma_{ab} \epsilon_J\,, \qquad \Phi^{IJ}{}_{ab} \equiv \bar{\epsilon}^I \gamma_{ab} \epsilon^J = (M_{IJ})^*\,, \tag{B.17}$$

   which is symmetric in the $SU(N)$ indices $\Phi_{IJ\,ab} = \Phi_{JI\,ab}$ and, further,

$${}^\star\Phi_{IJ\,ab} = -i\Phi_{IJ\,ab} \;\; \Rightarrow \;\; \Phi_{IJ\,ab} = \Phi_{IJ}{}^+{}_{ab}\,. \tag{B.18}$$

   As we are going to see, this matrix of 2-forms can be expressed entirely in terms of the scalar and vector bilinears.

It is straightforward to get identities for the products of these bilinears using the Fierz identity Eq. (B.8). First, the products of scalars:

$$M_{IJ}M_{KL} = \tfrac{1}{2}M_{IL}M_{KJ} - \tfrac{1}{8}\Phi_{IL}\cdot\Phi_{KJ}\,, \tag{B.19}$$

$$M_{IJ}M^{KL} = -\tfrac{1}{2}V^{L}{}_{I}\cdot V^{K}{}_{J}\,. \tag{B.20}$$

From Eq. (B.19) immediately follows

$$M_{I[J}M_{KL]} = 0\,, \tag{B.21}$$

which is a particular case of the Fierz identity

$$\epsilon_{[J}M_{KL]} = 0\,. \tag{B.22}$$

For $N = 4, 8, \ldots$, Eq. (B.21) implies, in turn

$$\mathrm{Pf}\, M = 0 \quad \Rightarrow \det M = 0\,. \tag{B.23}$$

For $N = 4$ we can define the $SU(4)$-dual of M_{IJ}

$$\tilde{M}_{IJ} \equiv \tfrac{1}{2}\varepsilon_{IJKL}M^{KL}\,, \qquad \varepsilon^{1234} = \varepsilon_{1234} = +1\,, \tag{B.24}$$

and the vanishing of the Pfaffian implies

$$\tilde{M}_{IJ}M^{IJ} = 0\,. \tag{B.25}$$

From Eq. (B.20) and the antisymmetry of M immediately follows

$$V^{I}{}_{L}\cdot V^{K}{}_{J} = -V^{I}{}_{J}\cdot V^{K}{}_{L} = -V^{K}{}_{L}\cdot V^{I}{}_{J}\,, \tag{B.26}$$

which implies that all the vector bilinears $V^{I}{}_{J\,a}$ are null:

$$V^{I}{}_{J}\cdot V^{I}{}_{J} = 0\,. \tag{B.27}$$

On the other hand, from Eqs. (B.26) and (B.20) follows the real $SU(N)$-invariant combination of vectors $V_a \equiv V^{I}{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^{I}{}_{J}\cdot V^{J}{}_{I} = 2M^{IJ}M_{IJ} \geq 0\,. \tag{B.28}$$

The products of M with the other bilinears[1] give

$$M_{IJ}V^{K}{}_{L\,a} = \tfrac{1}{2}M_{IL}V^{K}{}_{J\,a} + \tfrac{1}{2}\Phi_{IL\,ba}V^{K}{}_{J}{}^{b}\,, \tag{B.29}$$

$$M_{IJ}\Phi^{KL}{}_{ab} = V^{L}{}_{I\,[a|}V^{K}{}_{J\,|b]} - \tfrac{i}{2}\epsilon_{ab}{}^{cd}V^{L}{}_{I\,c}V^{K}{}_{J\,d}\,. \tag{B.30}$$

Now, let us consider the product of two arbitrary vectors[2]:

$$V^{I}{}_{J\,a}V^{K}{}_{L\,b} = \tfrac{i}{2}\epsilon_{ab}{}^{cd}V^{I}{}_{L\,c}V^{K}{}_{J\,d} + V^{I}{}_{L\,(a|}V^{K}{}_{J\,|b)} - \tfrac{1}{2}g_{ab}V^{I}{}_{L}\cdot V^{K}{}_{J}\,. \tag{B.31}$$

For V^2 this identity allows us to write the metric in the form

$$g_{ab} = 2V^{-2}[V_aV_b - V^{I}{}_{J\,a}V^{J}{}_{I\,b}]\,. \tag{B.32}$$

Following Tod [5], for $V^2 \neq 0$ we introduce

$$\mathcal{J}^{I}{}_{J} \equiv \frac{2M^{IK}M_{JK}}{|M|^2} = \frac{2V\cdot V^{I}{}_{J}}{V^2}\,, \qquad |M|^2 \equiv M^{LM}M_{LM} = \tfrac{1}{2}V^2\,. \tag{B.33}$$

Using Eq. (B.19) we can show that it is a Hermitean projector whose trace equals 2:

$$\mathcal{J}^{I}{}_{J}\mathcal{J}^{J}{}_{K} = \mathcal{J}^{I}{}_{K}\,, \qquad \mathcal{J}^{I}{}_{I} = +2\,. \tag{B.34}$$

Further, using the general Fierz identity we find

$$\mathcal{J}^{I}{}_{J}\epsilon^{J} = \epsilon^{I}\,, \qquad \epsilon_{I}\mathcal{J}^{I}{}_{J} = \epsilon_{J}\,, \tag{B.35}$$

which should be understood for $N > 2$ of the fact that the ϵ^I are not linearly independent[3]. As a consequence of the above identity, the contraction of $\mathcal{J}$ with any of the bilinears is the identity. Using this result and Eq. (B.30), we find

$$\Phi^{KL}{}_{ab} = \frac{2M^{IK}M_{IJ}}{|M|^2}\Phi^{JL}{}_{ab} = \frac{2M^{IK}}{|M|^2}V^{L}{}_{I\,[a}V_{b]} - i\frac{M^{IK}}{|M|^2}\epsilon_{ab}{}^{cd}V^{L}{}_{I\,c}V_d\,. \tag{B.36}$$

Other useful identities are

[1]We omit the product $M_{IJ}\Phi_{KL\,ab}$ which will not be used.
[2]The product $V^{I}{}_{J\,a}V_{L}{}^{K}{}_{b}$ gives a different identity that will not be used
[3]For $N = 2$ $\mathcal{J}^{I}{}_{J} = \delta^{I}{}_{J}$. See later on.

$$\frac{M_{IJ}M^{KL}}{|M|^2} = \mathcal{J}^K{}_{[I}\mathcal{J}^L{}_{J]}\,, \tag{B.37}$$

and

$$\frac{2\tilde{M}^{IK}\tilde{M}_{JK}}{|M|^2} = \delta^I{}_J - \mathcal{J}^I{}_J \equiv \tilde{\mathcal{J}}^I{}_J\,, \tag{B.38}$$

which is the complementary projector.

In the null case $V^2 = |M|^2 = 0$ it is customary to write $l_a \equiv V^I{}_{I\,a}$. Since $|M|^2$ is a sum of positive numbers, each of them must vanish independently, i.e. $M^{IJ} = 0$. This implies that all spinors ϵ^I are proportional and one can write

$$\epsilon_I = \phi_I \epsilon\,, \tag{B.39}$$

for some complex functions ϕ_I which transform as an $SU(4)$ vector, and some negative-chirality spinor ϵ. These are defined up to a rescaling by a complex function and opposite weights. Part of this freedom can be fixed by normalizing

$$\phi_I \phi^I = 1\,, \qquad \phi^I \equiv \phi_I^*\,. \tag{B.40}$$

Then, the only freedom that remains in the definition of ϕ^I is a change by a local phase $\theta(x)$

$$\phi_I \to e^{i\theta}\phi_I\,, \qquad \epsilon \to e^{-i\theta}\epsilon\,. \tag{B.41}$$

In this case on can construct another Hermitean projector $\mathcal{K}^I{}_J$ that plays a role analogous to that of $\mathcal{J}^I{}_J$ in the non-null case:

$$\mathcal{K}^I{}_J \equiv \phi^I \phi_J\,, \tag{B.42}$$

which satisfies

$$\mathcal{K}^I{}_J \mathcal{K}^J{}_K = \mathcal{K}^I{}_K\,, \qquad \mathcal{K}^I{}_I = +1\,, \tag{B.43}$$

and

$$\mathcal{K}^I{}_J \epsilon^J = \epsilon^I\,, \qquad \epsilon_I \mathcal{K}^I{}_J = \epsilon_J\,, \tag{B.44}$$

which expresses the known fact that only one spinor is linearly independent in this case.

In the null case, all the vector bilinears are also proportional to the null vector l:

$$V^I{}_{Ja} = \mathcal{K}^I{}_J l_a \,. \tag{B.45}$$

Once ϵ is given, we may introduce an auxiliary spinor with the same chirality and opposite $U(1)$ charge as ϵ and normalized against ϵ by

$$\bar{\epsilon}\eta = \frac{1}{2}\,, \tag{B.46}$$

where $\bar{\epsilon} = i\epsilon^T\gamma_0$. With both spinors we can construct a complex null tetrad with metric Eq. (A.27) as follows:

$$l_\mu = i\bar{\epsilon^*}\gamma_\mu\epsilon\,, \quad n_\mu = i\bar{\eta^*}\gamma_\mu\eta\,, \quad m_\mu = i\bar{\epsilon^*}\gamma_\mu\eta = i\bar{\eta}\gamma_\mu\epsilon^*\,, \quad m^*_\mu = i\bar{\epsilon}\gamma_\mu\eta^* = i\bar{\eta^*}\gamma_\mu\epsilon\,. \tag{B.47}$$

The normalization condition (B.40) does not fix completely the auxiliary spinor η and the freedom in the choice of η becomes a freedom in the null tetrad. First of all, there is a $U(1)$ freedom Eq. (B.41) under which $\eta' = e^{i\theta}\eta$ and

$$l' = l\,, \quad n' = n\,, \quad m' = e^{2i\theta}m\,. \tag{B.48}$$

Further, we can also shift η by terms proportional to ϵ preserving the normalization

$$\eta' = \eta + \delta\epsilon\,. \tag{B.49}$$

Under this redefinition of η, the null tetrad transforms as follows:

$$l' = l\,, \quad n' = n + \delta^* m + \delta m^* + |\delta|^2 l\,, \quad m' = m + \delta l\,. \tag{B.50}$$

B.1.3 The $N = 2$ case

Here we describe some of the peculiarities of the $N = 2$ case in which the number of spinors is precisely the necessary to construct a basis at each point.

In the $N = 2$ case there is only one independent (complex) scalar X since

$$\bar{\epsilon}_I\epsilon_J = X\epsilon_{IJ}\,, \tag{B.51}$$

where ϵ_{IJ} is the (constant) 2-dimensional totally antisymmetric tensor. It follows that

$$|M|^2 = 2|X|^2\,, \tag{B.52}$$

and, using $\epsilon_{IJ}\epsilon^{KL} = \delta_{IJ}{}^{KL}$ we can show that the projector

$$\mathcal{J}^I{}_J = \delta^I{}_J \, . \tag{B.53}$$

In the $|M|^2 \neq 0$ case, the four vector bilinears $V^I{}_{J\,\mu}$ can be used as a null tetrad

$$l_\mu = V^1{}_{1\,\mu}\,, \quad n_\mu = V^2{}_{2\,\mu}\,, \quad m_\mu = V^1{}_{2\,\mu}\,, \quad m^*_\mu = V^2{}_{1\,\mu}\,, . \tag{B.54}$$

Alternatively. one can use the four combinations

$$V^a{}_\mu \equiv \tfrac{1}{\sqrt{2}} V^I{}_{J\,\mu}(\sigma^a)^J{}_I\,, \tag{B.55}$$

with $\sigma^0 = 1$ and σ^i the three (traceless, Hermitean) Pauli matrices as an orthonormal tetrad in which V^0 is timelike and the V^i are spacelike.

B.1.4 The $U(2)$ formalism

We may always parametrize the $SU(4)$ spinors in terms of "$U(2)$ spinors" and a set of scalars:

$$\epsilon_I = \phi^A_I \epsilon_A\ , \quad A = 1, 2\ , \tag{B.56}$$

where ϕ^A_I is a vector of $SU(4)$ and ϵ_A is invariant, whereas ϕ^A_I is invariant under $SL(2,\mathbb{R})$ and ϵ_A has weight $+1$.

The above parametrization has a local $GL(2,\mathbb{C})$ symmetry. Part of this symmetry can be fixed by imposing the following normalization condition

$$\phi^A_I \phi^I_B = \delta^A_B\ , \tag{B.57}$$

where $\phi^I_A \equiv \phi^{A\,*}_I$. In general, to take the complex conjugate we raise and lower the $SU(4)$ and $U(2)$ indices. The above condition is preserved by $U(2)$,

$$\epsilon_A{}' = U_A{}^B \epsilon_B\ , \quad \phi^{A\prime}_I = \phi^B_I U^\dagger{}_B{}^A\ , \quad U \in U(2)\ . \tag{B.58}$$

We can also impose a further condition on ϕ^A in a $U(2)$-covariant way. Let X be the $U(2)$ density

$$X \equiv \frac{1}{2|M|} M_{IJ} \phi^I_A \phi^J_B \epsilon^{AB}\ , \tag{B.59}$$

after a $U(2)$ transformation X transform as

$$X' = \det U X\ . \tag{B.60}$$

Then we fix the modulus of X in the following way

$$i\sqrt{2}X = e^{i\lambda} \tag{B.61}$$

where λ is an arbitrary scalar. Therefore we have

$$|X|^2 = \frac{1}{2} \, . \tag{B.62}$$

Note that X has $SL(2,\mathbb{R})$ weight $+2$ and consequently λ transforms under $SL(2,\mathbb{R})$ as

$$\lambda' = \lambda + \alpha \, . \tag{B.63}$$

On the other hand, the determinant of an arbitrary $U(2)$ transformation is an arbitrary phase. Therefore the effects of $SL(2,\mathbb{R})$ and $U(2)$ transformations on X and λ are the same.

There is a $U(2)$ connection given by

$$\phi^I_A d\phi^B_I \tag{B.64}$$

which is an anti-hermitean matrix in the $U(2)$ indices. It is useful to decompose this connection in its trace and traceless parts,

$$\zeta = \phi^I_A d\phi^A_I \, , \tag{B.65}$$

$$\mathcal{A}_A{}^B = \phi^I_A d\phi^B_I - \frac{1}{2}\delta_A{}^B \zeta \tag{B.66}$$

such that ζ is a imaginary one-form and $\mathcal{A}$ is a traceless anti-hermitean matrix-valued one-form. ζ is a gauge connection for the $U(1)$ part of $U(2)$ given by the trace generator (the identity) and $\mathcal{A}$ is a gauge connection for the $SU(2)$ part given by the traceless generators (the Pauli matrices). The curvature of $\mathcal{A}$ is given by

$$\mathcal{R}_A{}^B = d\mathcal{A}_A{}^B - \mathcal{A}_A{}^C \wedge \mathcal{A}_C{}^B \, . \tag{B.67}$$

Let $V_A{}^B$ be four vectors constructed as bilinears of the $U(2)$ spinors,

$$V^{aB}_A \equiv i\bar{\epsilon}_A \gamma^a \epsilon^B \, . \tag{B.68}$$

The $U(2)$ vectors are related to the $SU(4)$ vectors by

$$V_I{}^J = \phi^A_I V_A{}^B \phi^J_B \, , \quad V_A{}^B = \phi^I_A V_I{}^J \phi^B_J \, . \tag{B.69}$$

In particular the trace vector of $U(2)$ is equal to V. The $SU(4)$ vectors satisfy the following (Fierz) identity

$$|M|^2 g_{\mu\nu} = V_\mu V_\nu - V_{\mu\, I}{}^J V_{\nu\, J}{}^I \, , \tag{B.70}$$

such that

$$|M|^2 g_{\mu\nu} = V_\mu V_\nu - V_{\mu\, A}{}^B V_{\nu\, B}{}^A \, . \tag{B.71}$$

We may decompose the $U(2)$ vectors as follows

$$V_A{}^B = \frac{1}{2}\delta_A{}^B V + \frac{1}{\sqrt{2}}\sigma^{x\,B}_A V_x\ , \quad x = 1,2,3. \tag{B.72}$$

which is equivalent to pass from the fundamental×antifundamental representation of $U(2)$ to the adjoint one. Indeed, the inverse transformations for the $SU(2)$ sector are

$$V^x = \frac{1}{\sqrt{2}}\sigma^{x\,B}_A V_B{}^A\ . \tag{B.73}$$

Then the (t,t) component of the equation (B.71) yields

$$V_t^x = 0 \tag{B.74}$$

and the spatial components are

$$h_{ij} = V_i^x V_j^y \delta_{xy}\ , \tag{B.75}$$

where $i, j, k...$ are curved spatial indices. Therefore the three vectors V^x, which are time-independent and have not time component, are vielbeins for the spatial metric h_{ij}. We may introduce objects of the tangent space using the V^x basis. For instance the spin connection is introduced by

$$\nabla_i V_j{}^x = \partial_i V_j{}^x - \Gamma_{ij}{}^k V_k{}^x - o_{iy}{}^x V_j{}^y \tag{B.76}$$

B.2 Five dimensions

B.2.1 Gamma matrices and spinors

The first four of our 5-dimensional gamma matrices are taken to be identical to 4-dimensional purely imaginary gamma matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ satisfying

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\,, \tag{B.77}$$

and the fifth is $\gamma^4 = -\gamma^{0123}$, so it is purely real, the above anticommutator is valid for $a = 0, \cdots, 4$ and, furthermore, $\gamma^{a_1\cdots a_5} = +\varepsilon^{a_1\cdots a_5}$ and, in general

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]}}{(5-n)!}\varepsilon^{a_1\cdots a_n b_1\cdots b_{n-5}}\gamma_{b_1\cdots b_{n-5}}\,. \tag{B.78}$$

On the other hand, γ^0 is Hermitean and the other gammas are anti-Hermitean.

To explain our convention for symplectic-Majorana spinors, let us start by defining the Dirac, complex and charge conjugation matrices $\mathcal{D}_\pm, \mathcal{B}_\pm, \mathcal{C}_\pm$. By definition, they satisfy

$$\mathcal{D}_\pm \gamma^a \mathcal{D}_\pm^{-1} = \pm\gamma^{a\,\dagger}\,, \qquad \mathcal{B}_\pm \gamma^a \mathcal{B}_\pm^{-1} = \pm\gamma^{a\,*}\,. \qquad \mathcal{C}_\pm \gamma^a \mathcal{C}_\pm^{-1} = \pm\gamma^{a\,T}\,. \tag{B.79}$$

The natural choice for Dirac conjugation matrix is

$$\mathcal{D} = i\gamma^0\,, \tag{B.80}$$

which corresponds to $\mathcal{D} = \mathcal{D}_+$. The other conjugation matrices are related to it by

$$\mathcal{C}_\pm = \mathcal{B}_\pm^T \mathcal{D}\,, \tag{B.81}$$

but it can be shown that in this case only $\mathcal{C} = \mathcal{C}_+$ and $\mathcal{B} = \mathcal{B}_+$ exist and are both antisymmetric. We take them to be

$$\mathcal{C} = i\gamma^{04}\,, \qquad \mathcal{B} = \gamma^4 \quad \Rightarrow \mathcal{B}^*\mathcal{B} = -1\,. \tag{B.82}$$

The Dirac conjugate is defined by

$$\psi^\dagger \mathcal{D} = i\psi^\dagger\gamma^0\,, \tag{B.83}$$

and the Majorana conjugate by

$$\psi^T \mathcal{C} = i\psi^T\gamma^{04}\,. \tag{B.84}$$

The Majorana condition (Dirac conjugate = Majorana conjugate) cannot be consistently imposed because it requires $\mathcal{B}^*\mathcal{B} = +1$. Therefore, we introduce the symplectic-Majorana conjugate in pairs of spinors by using the corresponding symplectic matrix, *e.g.*

$$\psi^{i\,c} \equiv \varepsilon_{ij}\psi^{j\,T}\mathcal{C}\,, \tag{B.85}$$

then the symplectic-Majorana condition is

$$\psi^{i\,*} = \varepsilon_{ij}\gamma^4\psi^j\,. \tag{B.86}$$

To impose the symplectic-Majorana condition on hyperinos ζ^A the only thing we have to do is to replace the matrix ε_{ij} by $\mathbb{C}_{AB}$, which is the invariant metric of $Sp(n_h)$.

Our conventions on $SU(2)$ indices are intended to keep manifest the $SU(2)$ covariance. In $SU(2)$, besides the preserved metric, there is the preserved tensor ε_{ij}. We also introduce ε^{ij}, $\varepsilon_{12} = \varepsilon^{12} = +1$. Therefore we may construct new covariant objects by using ε_{ij} and ε^{ij}, for instance $\psi_i \equiv \varepsilon_{ij}\psi^j$ (whence $\psi^j = \psi_i\varepsilon^{ij}$). With this notation the symplectic-Majorana condition can be simply stated as

$$\psi^{i\,*} = \gamma^4\psi_i \,. \tag{B.87}$$

We use the bar on spinors to denote the (single) Majorana conjugate:

$$\bar{\psi}^i \equiv {\psi^i}^T\mathcal{C} \,, \tag{B.88}$$

which transforms under $SU(2)$ in the same representation as ψ^i does. We also lower its $SU(2)$ index: $\bar{\psi}_i \equiv \varepsilon_{ij}\bar{\psi}^j$. In terms of single Majorana conjugates the symplectic Majorana condition reads

$$\left(\bar{\psi}^i\right)^* = \bar{\psi}_i\gamma^4 \,. \tag{B.89}$$

Finally, observe that after imposing the symplectic Majorana condition the following simple relation between the single Dirac and Majorana conjugates holds:

$${\psi^i}^\dagger\mathcal{D} = \bar{\psi}_i \,, \tag{B.90}$$

which is very useful if one prefers to use the Dirac conjugate instead of the Majorana one.

The bilinears that can be constructed from Killing spinors will in general be 2×2 matrices that can be written as linear combinations of the Pauli matrices $\sigma^{\hat{r}}$ ($\hat{r} = 0,\ldots,3$) where $\sigma^0 = \mathbb{I}_{2\times 2}$. Therefore, we are bound to need the Fierz identities

$$\begin{aligned}\left(\bar{\lambda}M\varphi\right)\left(\bar{\psi}N\chi\right) \;=\;& \tfrac{p}{8}\left\{\left(\bar{\lambda}M\sigma^{\hat{r}}N\chi\right)\left(\bar{\psi}\sigma^{\hat{r}}\varphi\right) + \left(\bar{\lambda}M\gamma^a\sigma^{\hat{r}}N\chi\right)\left(\bar{\psi}\gamma_a\sigma^{\hat{r}}\varphi\right)\right.\\ &\left.-\tfrac{1}{2}\left(\bar{\lambda}M\gamma^{ab}\sigma^{\hat{r}}N\chi\right)\left(\bar{\psi}\gamma_{ab}\sigma^{\hat{r}}\varphi\right)\right\} \,,\end{aligned} \tag{B.91}$$

where the $SU(2)$ indices are implicit and $p = (-)1$ for (anti-)commuting spinors.

B.2.2 Spinor bilinears

With one commuting symplectic-Majorana spinor ϵ^i we can construct the following independent, $SU(2)$-covariant bilinears:

$\bar{\epsilon}_i\ \epsilon^j$: It is easy to see that

$$\begin{aligned} \bar{\epsilon}_i \epsilon^j &= -\varepsilon^{jk}(\bar{\epsilon}_k \epsilon^l)\varepsilon_{li}\,, \\ (\bar{\epsilon}_i \epsilon^j)^* &= -\bar{\epsilon}_j \epsilon^i\,, \end{aligned} \tag{B.92}$$

The first equation implies that this matrix is proportional to $\delta_i{}^j$ and the second equation implies that the constant is purely imaginary. Thus, we define the $SU(2)$-invariant scalar

$$f \equiv i\bar{\epsilon}_i\epsilon^i = i\bar{\epsilon}\sigma^0\epsilon \quad , \quad \bar{\epsilon}_i\epsilon^j = -\tfrac{i}{2}\, f\ \delta_i{}^j\ . \tag{B.93}$$

All the other scalar bilinears $i\bar{\epsilon}\sigma^r\epsilon$ $(r = 1, 2, 3)$ vanish identically.

$\bar{\epsilon}_i\gamma^a\epsilon^j$: This matrix satisfies the same properties as $\bar{\epsilon}_i\epsilon^j$, and so we define the vector bilinear

$$V^a \equiv i\bar{\epsilon}_i\gamma^a\epsilon^i = i\bar{\epsilon}\gamma^a\sigma^0\epsilon \quad , \quad \bar{\epsilon}_i\gamma^a\epsilon^j = -\tfrac{i}{2}\ \delta_i{}^j\ V^a\ . \tag{B.94}$$

which is also $SU(2)$-invariant, the other vector bilinears being automatically zero.

$\bar{\epsilon}_i\gamma^{ab}\epsilon^j$: In this case

$$\begin{aligned} \bar{\epsilon}_i\gamma^{ab}\epsilon^j &= +\varepsilon^{jk}(\bar{\epsilon}_k\gamma^{ab}\epsilon^l)\varepsilon_{li}\,, \\ (\bar{\epsilon}_i\gamma^{ab}\epsilon^j)^* &= \bar{\epsilon}_j\gamma^{ab}\epsilon^i\,, \end{aligned} \tag{B.95}$$

which means that these 2-form matrices are traceless and Hermitean and we have three non-vanishing real 2-forms

$$\Phi^{r\,ab} \equiv \sigma^r{}_i{}^j\ \bar{\epsilon}_j\gamma^{ab}\epsilon^i \quad , \quad \bar{\epsilon}_i\gamma^{ab}\epsilon^j = \tfrac{1}{2}\sigma^r{}_i{}^j\ \Phi^{r\,ab}\ . \tag{B.96}$$

$r = 1, 2, 3$, which transform as a vector in the adjoint representation of $SU(2)$, and the fourth $\bar{\epsilon}\gamma^{ab}\sigma^0\epsilon = 0$.

Using the Fierz identities Eq. (B.91) for commuting spinors we get, among other identities,

$$V^a V_a = f^2 \,, \tag{B.97}$$

$$V_a V_b = \eta_{ab} f^2 + \tfrac{1}{3}\Phi^r{}_a{}^c \Phi^r{}_{cb} \,, \tag{B.98}$$

$$V^a \Phi^r{}_{ab} = 0 \,, \tag{B.99}$$

$$V^a ({}^\star\Phi^r)_{abc} = -f \Phi^r{}_{bc} \,, \tag{B.100}$$

$$\Phi^r{}_a{}^c \Phi^s{}_{cb} = -\delta^{rs}(\eta_{ab} f^2 - V_a V_b) - \varepsilon^{rst} f \Phi^t{}_{ab} \,, \tag{B.101}$$

$$\Phi^r{}_{[ab} \Phi^s{}_{cd]} = -\tfrac{1}{4} f \delta^{rs} \varepsilon_{abcde} V^e \,, \tag{B.102}$$

$$V_a \gamma^a \epsilon^i = f \epsilon^i \,, \tag{B.103}$$

$$\Phi^r_{ab} \gamma^{ab} \epsilon^i = 4 i f \epsilon^j \sigma^r{}_j{}^i \,. \tag{B.104}$$

C

Supersymmetric space-time metrics

C.1 Four dimensional conformastationary metric

A conformastationary metric has the general form

$$ds^2 = |M|^2(dt + \omega)^2 - |M|^{-2}\gamma_{\underline{ij}}dx^i dx^j\,, \qquad i, j = 1, 2, 3\,, \tag{C.1}$$

where all components of the metric are independent of the time coordinate t. Choosing the Vielbein basis

$$(e_\mu{}^a) = \begin{pmatrix} |M| & |M|\omega_{\underline{i}} \\ & \\ 0 & |M|^{-1}v_{\underline{i}}{}^j \end{pmatrix}, \qquad (e_a{}^\mu) = \begin{pmatrix} |M|^{-1} & -|M|\omega_i \\ & \\ 0 & |M|v_i{}^{\underline{j}} \end{pmatrix}, \tag{C.2}$$

where

$$\gamma_{\underline{ij}} = v_{\underline{i}}{}^k v_{\underline{j}}{}^l \delta_{kl}\,, \qquad v_i{}^{\underline{k}} v_{\underline{k}}{}^j v_j\,, \qquad \omega_i = v_i{}^{\underline{j}}\omega_{\underline{j}}\,, \tag{C.3}$$

we find that the spin connection components are

$$\begin{aligned} \omega_{00i} &= -\partial_i|M|\,, & \omega_{0ij} &= \tfrac{1}{2}f_{ij}\,, \\ \omega_{i0j} &= \omega_{0ij}\,, & \omega_{ijk} &= -|M|o_{ijk} - 2\delta_{i[j}\partial_{k]}|M|\,, \end{aligned} \tag{C.4}$$

where $o_i{}^{jk}$ is the 3-dimensional spin connection and

$$\partial_i \equiv v_i{}^{\underline{j}}\partial_{\underline{j}}\,, \qquad f_{ij} = v_i{}^{\underline{k}}v_j{}^{\underline{l}}f_{\underline{kl}}\,, \qquad f_{\underline{ij}} \equiv 2\partial_{[\underline{i}}\omega_{\underline{j}]}\,. \tag{C.5}$$

The components of the Riemann tensor are

$$\begin{aligned} R_{0i0j} &= \tfrac{1}{2}\nabla_i\partial_j|M|^2 + \partial_i|M|\partial_j|M| - \delta_{ij}(\partial|M|)^2 + \tfrac{1}{4}\nabla i|M|^6 f_{ik}f_{jk}\,, \\ R_{0ijk} &= -\tfrac{1}{2}\nabla_i(|M|^4 f_{jk}) + \tfrac{1}{2}f_{i[j}\partial_{k]}|M|^4 - \tfrac{1}{4}\delta_{i[j}f_{k]l}\partial_l|M|^4\,, \\ R_{ijkl} &= -|M|^2 R_{ijkl} + \tfrac{1}{2}|M|^6(f_{ij}f_{kl} - f_{k[i}f_{j]l}) - 2\delta_{ij,kl}(\partial|M|)^2 + 4|M|\delta_{[i}{}^{[k}\nabla_{j]}\partial^{l]}|M|\,, \end{aligned} \tag{C.6}$$

where all the objects in the right-hand sides of the equations are referred to the 3-dimensional spatial metric. The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -|M|^2\nabla^2\log|M| - \tfrac{1}{4}|M|^6 f^2\,, \\ R_{0i} &= \tfrac{1}{2}\nabla_j(|M|^4 f_{ji})\,, \\ R_{ij} &= |M|^2\{R_{ij} + 2\partial_i\log|M|\partial_j\log|M| - \delta_{ij}\nabla^2\log|M| - \tfrac{1}{2}|M|^4 f_{ik}f_{jk}\}\,, \end{aligned} \tag{C.7}$$

and the Ricci scalar is

$$R = -|M|^2\{R - \tfrac{1}{4}|M|^4 f^2 - 2\nabla^2\log|M| + 2(\partial\log|M|)^2\}\,, \tag{C.8}$$

C.2 Four dimensional Brinkmann pp-wave metric

These metrics are

$$ds^2 = 2du(dv + Kdu + \omega) - 2e^{2U}dzdz^*\,, \qquad \omega = \omega_{\underline{z}}dz + \omega_{\underline{z}^*}dz^*\,, \tag{C.9}$$

where all the functions in the metric are independent of v.

Using also light-cone coordinates in tangent space, a natural Vielbein basis is

$$
\begin{array}{rclclrclcl}
e^u & = & du & = & \hat{l}\,, & e_u & = & \partial_{\underline{u}} - K\partial_{\underline{v}} & = & n^\mu \partial_\mu\,, \\
e^v & = & dv + Kdu + \omega & = & \hat{n}\,, & e_v & = & \partial_{\underline{v}} & = & l^\mu \partial_\mu\,, \\
e^z & = & e^U dz & = & \hat{m}\,, & e_z & = & e^{-U}(\partial_{\underline{z}} - \omega_{\underline{z}}\partial_{\underline{v}}) & = & -m^{*\,\mu}\partial_\mu\,, \\
e^{z^*} & = & e^U dz^* & = & \hat{m}^*\,, & e_{z^*} & = & e^{-U}(\partial_{\underline{z}^*} - \omega_{\underline{z}^*}\partial_{\underline{v}}) & = & -m^\mu \partial_\mu\,.
\end{array}
\tag{C.10}
$$

The components of the spin connection are

$$
\begin{array}{rclrcl}
\omega_{uzu} & = & e^{-U}(\partial_{\underline{z}} K - \dot{\omega}_{\underline{z}})\,, & \omega_{uzz^*} & = & \frac{1}{2}e^{-2U} f_{\underline{zz}^*} - \dot{U}\,, \\
\omega_{zz^*u} & = & -\frac{1}{2}e^{-2U} f_{\underline{zz}^*} - \dot{U}\,, & \omega_{zzz^*} & = & -e^{-U}\partial_{\underline{z}} U\,,
\end{array}
\tag{C.11}
$$

where $f_{\underline{zz}^*} = 2\partial_{[\underline{z}}\omega_{\underline{z}^*]}$ and a dot stands for partial derivation with respect to u.

The components of the Ricci tensor are

$$
\begin{array}{rcl}
R_{zz^*} & = & 2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*} U\,, \\
R_{zu} & = & \frac{1}{2}e^{-3U}\partial_{\underline{z}} f_{\underline{zz}^*} + e^{-U}(\partial_{\underline{z}}\dot{U} + \dot{U}\partial_{\underline{z}} U)\,, \\
R_{uu} & = & -2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*} K + \frac{1}{2}(f_{\underline{zz}^*})^2 + e^{-2U}(\partial_{\underline{z}}\dot{\omega}_{\underline{z}^*} + \partial_{\underline{z}^*}\dot{\omega}_{\underline{z}}) + 2(\ddot{U} + \dot{U}\dot{U})\,,
\end{array}
\tag{C.12}
$$

and the Ricci scalar is just

$$
R = -4e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*} U\,. \tag{C.13}
$$

C.3 The five-dimensional time-like metric

In the timelike case we find the conformastationary metric

$$
ds^2 = f^2\,(dt + \omega)^2 - f^{-1} h_{\underline{mn}} dx^m dx^n\,, \qquad \omega = \omega_{\underline{m}} dx^m\,, \qquad m,n = 1,\cdots,4\,. \tag{C.14}
$$

We choose the Vielbein basis

$$(e^a{}_\mu) = \begin{pmatrix} f & f\omega_{\underline{m}} \\ 0 & f^{-1/2}V^n{}_{\underline{m}} \end{pmatrix}, \qquad (e^\mu{}_a) = \begin{pmatrix} f^{-1} & -f^{1/2}\omega_m \\ 0 & f^{1/2}V^{\underline{n}}{}_m \end{pmatrix}, \tag{C.15}$$

where

$$h_{\underline{mn}} = V_{\underline{m}}{}^p V_{\underline{n}}{}^q \delta_{pq}\,, \qquad V_m{}^{\underline{p}} V_n{}^{\underline{q}} h_{\underline{pq}} = \delta_{mn}\,, \qquad \omega_m = V_m{}^{\underline{n}}\omega_{\underline{n}}\,. \tag{C.16}$$

The non-vanishing components of the spin connection in this basis are

$$\omega_{00m} = -2\partial_m f^{1/2}\,, \quad \omega_{0mn} = \omega_{m0n} = \tfrac{1}{2}f^2\,(d\omega)_{mn}\,, \quad \omega_{mnp} = -f^{1/2}\xi_{mnp} - 2\delta_{m[n}\partial_{p]}f^{1/2}\,, \tag{C.17}$$

where, from now on, all the objects in the r.h.s. of these equations refer to the 4-dimensional metric $h_{\underline{mn}}$ and, in particular

$$(d\omega)_{mn} = V_m{}^{\underline{p}} V_n{}^{\underline{q}}\,(d\omega)_{\underline{pq}} = 2V_m{}^{\underline{p}} V_n{}^{\underline{q}}\partial_{[\underline{p}}\omega_{\underline{q}]}\,. \tag{C.18}$$

Thee non-vanishing components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -\nabla^2 f + f^{-1}(\partial f)^2 - \tfrac{1}{4}f^4(d\omega)^2\,, \\ R_{0m} &= -\tfrac{1}{2}f^{-1/2}\nabla_n[f^3(d\omega)_{nm}]\,, \\ R_{mn} &= fR_{mn} - \tfrac{1}{2}(d\omega)_{mp}(d\omega)_{np} + \tfrac{3}{2}f^{-1}\partial_m f\partial_n f - \tfrac{1}{2}\delta_{mn}[\nabla^2 f - f^{-1}(\partial f)^2]\,, \end{aligned} \tag{C.19}$$

and the Ricci scalar is given by

$$R = -fR + \tfrac{1}{4}(d\omega)^2 + \nabla^2 f - \tfrac{5}{2}f^{-1}(\partial f)^2\,. \tag{C.20}$$

C.4 The five-dimensional null case metric

$$ds^2 = 2fdu(dv + Hdu + \omega) - f^{-2}\gamma_{\underline{rs}}dx^r dx^s\,, \qquad r,s = 1,2,3\,. \tag{C.21}$$

Orthonormal 1-form and vector basis for this metric are given by

$$
\begin{aligned}
e^{+} &= f du\,, & e_{+} &= f^{-1}(\partial_{\underline{u}} - H\partial_{\underline{v}})\,,\\
e^{-} &= dv + H du + \omega\,, & e_{-} &= \partial_{\underline{v}}\,,\\
e^{r} &= f^{-1}v^{r}\,, & e_{r} &= f(v_{r} - \omega_{r}\partial_{\underline{v}})\,,
\end{aligned}
\tag{C.22}
$$

where $v^r = v^r{}_{\underline{s}} dx^s$ and $v_r = v_r{}^{\underline{s}}\partial_{\underline{s}}$ are orthonormal basis 1-forms and vectors for the 3-dimensional spatial positive-definite metric $\gamma_{\underline{rs}}$

$$
\delta_{rs} v^r{}_{\underline{t}} v^s{}_{\underline{q}} = \gamma_{\underline{tq}}\,, \qquad v_t{}^{\underline{r}} v_q{}^{\underline{s}} \gamma_{\underline{rs}} = \delta_{tq}\,. \tag{C.23}
$$

The non-vanishing components of the spin connection are

$$
\begin{aligned}
\omega_{+r+} &= \partial_r H - \partial_{\underline{u}}\omega_{\underline{s}} v_r{}^{\underline{s}}\,, & \omega_{rs+} &= -\tfrac{1}{2} f^2 F_{rs} - f^{-2}\partial_{\underline{u}} f \delta_{rs} - f^{-1} v_{(r|}{}^{\underline{t}}\partial_{\underline{u}} v_{|s)\underline{t}}\,,\\
\omega_{+r-} &= \tfrac{1}{2}\partial_r f = \omega_{-r+} = -\omega_{r+-}\,, & \omega_{+rs} &= \tfrac{1}{2} f^2 F_{rs} - f^{-1} v_{[r|}{}^{\underline{t}}\partial_{\underline{u}} v_{|s]\underline{t}}\,,\\
\omega_{rst} &= f\varpi_{rst} - 2\delta_{r[s}\partial_{t]} f\,,
\end{aligned}
\tag{C.24}
$$

where all the quantities in the r.h.s. of all these equations refer to the 3-dimensional metric and Dreibein and

$$
F_{rs} = v_r{}^{\underline{t}} v_s{}^{\underline{p}} F_{\underline{tp}}\,, \qquad F_{\underline{rs}} \equiv 2\partial_{[\underline{r}}\omega_{\underline{s}]}\,. \tag{C.25}
$$

The non-vanishing components of the Ricci tensor are

$$
\begin{aligned}
R_{++} &= -f\nabla^2 H - \tfrac{1}{4} f^4 F^2 + f\nabla^{\underline{r}}\dot{\omega}_{\underline{r}} + 3\dot{\omega}_{\underline{r}}\partial^{\underline{r}} f + \tfrac{1}{2} f^{-2}\gamma^{\underline{rs}}\ddot{\gamma}_{\underline{rs}} + \tfrac{1}{4} f^{-2}\dot{\gamma}^{\underline{rs}}\dot{\gamma}_{\underline{rs}}\\
&\quad -\tfrac{3}{2} f^{-3}\dot{f}\gamma^{\underline{rs}}\dot{\gamma}_{\underline{rs}} - 3f^{-2}\left[\partial_{\underline{u}}^2 \log f - 2\left(\partial_{\underline{u}}\log f\right)^2\right]\,,\\
R_{+-} &= -\tfrac{1}{2} f^2\nabla^2 \log f\,,\\
R_{+r} &= -\tfrac{1}{2}\nabla_s\left(f^3 F_{sr}\right) - \tfrac{1}{2} v_r{}^{\underline{r}}\gamma^{\underline{st}}\nabla_{\underline{s}}\dot{\gamma}_{\underline{rt}} + \tfrac{1}{2} v_r{}^{\underline{r}}\partial_{\underline{u}}\left(\gamma^{\underline{st}}\partial_{\underline{r}}\gamma_{\underline{st}}\right) + \tfrac{3}{2} v_r{}^{\underline{r}}\dot{\gamma}_{\underline{rt}}\partial^{\underline{t}}\log f\\
&\quad -\tfrac{3}{2}\partial_r\partial_{\underline{u}}\log f - \tfrac{3}{4}\gamma^{\underline{st}}\dot{\gamma}_{\underline{st}}\partial_r\log f + \tfrac{3}{2}\partial_{\underline{u}}\log f\,\partial_r\log f\,,\\
R_{rs} &= f^2 R_{rs}(\gamma) - \delta_{rs} f^2\nabla^2\log f + \tfrac{3}{2}\partial_r f\partial_s f\,,
\end{aligned}
\tag{C.26}
$$

and the Ricci scalar is

$$R = -f^2 R(\gamma) + 2f^2 \nabla^2 \log f - \tfrac{3}{2} \left(\partial f\right)^2 \ . \tag{C.27}$$

D

Scalar manifolds

D.1 Real Special Geometry

The geometry of the n physical scalars ϕ^x $(x = 1, \dots, n)$ of the vector multiplets is fully determined by a constant real symmetric tensor C_{IJK} $(I, J, K = 0, 1, \dots, \bar{n} \equiv n+1)$. The scalars appear through $\bar{n}$ functions $h^I(\phi)$ constrained to satisfy

$$C_{IJK} h^I h^J h^K = 1\,. \tag{D.1}$$

One defines

$$h_I \equiv C_{IJK} h^J h^K\,, \quad \Rightarrow h_I h^I = 1\,, \tag{D.2}$$

and a metric a_{IJ} that can be use to raise and lower the $SO(\bar{n})$ index

$$h_I \equiv a_{IJ} h^J\,, \qquad h^I \equiv a^{IJ} h_J\,. \tag{D.3}$$

The definition of h_I allows us to find

$$a_{IJ} = -2 C_{IJK} h^K + 3 h_I h_J\,. \tag{D.4}$$

Next, one defines

$$h^I_x \equiv -\sqrt{3} h^I{}_{,x} \equiv -\sqrt{3} \frac{\partial h^I}{\partial \phi^x}\,, \tag{D.5}$$

and

$$h_{Ix} \equiv a_{IJ} h^J_x = +\sqrt{3} h_{I,x}\,, \tag{D.6}$$

which satisfy

$$h_I h^I_x = 0\,, \qquad h^I h_{Ix} = 0\,, \tag{D.7}$$

due to Eq. (D.1). The h^I enjoy the following properties of closure and orthogonality

$$\begin{pmatrix} h^I \\ h^I_x \end{pmatrix} \begin{pmatrix} h_I & h^y_I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta^y_x \end{pmatrix}\,, \quad \begin{pmatrix} h_I & h^x_I \end{pmatrix} \begin{pmatrix} h^J \\ h^J_x \end{pmatrix} = \delta^J_I\,. \tag{D.8}$$

Therefore any object with $SO(\bar{n})$ index can be decomposed as

$$A^I = \left(h_J A^J\right) h^I + \left(h^x_J A^J\right) h^I_x\,. \tag{D.9}$$

The metric of the scalars $g_{xy}(\phi)$ is the pullback of a_{IJ}:

$$g_{xy} = a_{IJ} h^I_x h^J_y = -2 C_{IJK} h^I_x h^J_y h^K\,, \tag{D.10}$$

and can be used to raise and lower x, y indices. Other useful expressions are

$$a_{IJ} = h_I h_J + h^x_I h_{Jx}\,, \tag{D.11}$$

$$C_{IJK} h^K = h_I h_J - \tfrac{1}{2} h^x_I h_{Jx}\,, \tag{D.12}$$

and

$$h_I h_J = \tfrac{1}{3} a_{IJ} + \tfrac{2}{3} C_{IJK} h^K\,, \tag{D.13}$$

$$h^x_I h_{Jx} = \tfrac{2}{3} a_{IJ} - \tfrac{2}{3} C_{IJK} h^K\,. \tag{D.14}$$

We now introduce the Levi-Cività covariant derivative associated to the scalar metric g_{xy}

$$h_{Ix;y} \equiv h_{Ix,y} - \Gamma_{xy}{}^z h_{Iz}\,. \tag{D.15}$$

It can be shown that

$$h_{Ix;y} = \frac{1}{\sqrt{3}}(h_I g_{xy} + T_{xyz} h_I^z)\,, \tag{D.16}$$

$$h^I_{x;y} = -\frac{1}{\sqrt{3}}(h^I g_{xy} + T_{xyz} h^{Iz})\,, \tag{D.17}$$

$$T_{xyz} = \sqrt{3} h_{Ix;y} h^I_z = -\sqrt{3} h_{Ix} h^I_{y;z}\,, \tag{D.18}$$

$$\Gamma_{xy}{}^z = h^{Iz} h_{Ix,y} - \frac{1}{\sqrt{3}} T_{xy}{}^w = 8 h_I^z h^I_{x,y} + \frac{1}{\sqrt{3}} T_{xy}{}^w\,. \tag{D.19}$$

D.2 Quaternionic-Kähler manifolds

In this appendix we review the definition and basics of quaternionic-Kähler manifolds. We refer the reader to Ref. [116] for a more comprehensive introduction to quaternionic manifolds with original references.

A *quaternionic-Kähler manifold* is a real $4n$-dimensional manifold ($n > 1$) such that[1]

1. There exists on it a triplet of complex structures $J^r{}_X{}^Y$, $r = 1, 2, 3$, $X, Y = 1, \ldots 4n$ which satisfy the algebra of imaginary unit quaternions,

$$J^r J^s = -\delta^{rs} + \varepsilon^{rst} J^t\,, \tag{D.20}$$

   which is known as *hypercomplex or quaternionic structure*. A manifold with this property is an *almost hypercomplex* of almost *quaternionic manifold*.

2. The hypercomplex structure is integrable, *i.e.* it is covariantly constant with respect to the standard Levi-Cività connection and a non-trivial $\mathfrak{su}(2)$ connection (*i.e.* with non-vanishing curvature):

$$\partial_X J^r{}_Y{}^Z - \Gamma_{XY}{}^U J^r{}_U{}^Z + \Gamma_{XU}{}^Z J^r{}_Y{}^U + 2\varepsilon^{rst} \omega_X{}^s J^t{}_Y{}^Z = 0\,, \tag{D.21}$$

   where $\omega_X{}^r$ is the $\mathfrak{su}(2)$ connection. In this case the manifold is a *quaternionic manifold*. (If this equation is satisfied with a trivial $\mathfrak{su}(2)$ connection the manifold is a *hypercomplex manifold*.)

[1] Clearly, the definitions given below are just too weak to be useful when $n = 1$, and one defines a 4-dimensional manifold to be quaternionic-Kähler, iff it is Einstein and selfdual. For a supergravity justification of this definition see *e.g.* [116].

3. There is a metric which is invariant under the action of the three complex structures

$$g_{XY} = J^{(r)}{}_X{}^Z J^{(r)}{}_Y{}^U g_{ZU}\,, \qquad \text{(no sum over } r!)\,. \tag{D.22}$$

This property makes it a (quaternionic) Kähler manifold.

The combination of the complex structures with the metric gives us the three hyper-Kähler 2-forms

$$J^r{}_{XY} = g_{XZ} J^r{}_Y{}^Z\,. \tag{D.23}$$

They are covariantly closed respect to the $\mathfrak{su}(2)$ connection,

$$dJ^r + 2\varepsilon^{rst}\omega^s \wedge J^t = 0\,. \tag{D.24}$$

The holonomy of a quaternionic-Kähler manifold is contained in $SU(2)\cdot Sp(2)$ and the tangent space indices are split accordingly into pairs of $SU(2)$ and $Sp(n)$ indices $i,j,k = 1,2$ and $A,B,C = 1,\ldots,2n$ respectively. The Vielbein is defined to be $f_{iA}{}^X$ and is related to the metric by

$$g_{XY} \;=\; f_X{}^{iA}\, f_Y{}^{jB}\, \mathbb{C}_{AB}\, \varepsilon_{ij}\,, \tag{D.25}$$

where

$$f_X{}^{iA}\, f_{iA}{}^Y \;=\; \delta_X{}^Y\,, \qquad f_{iA}{}^X\, f_X{}^{jB} \;=\; \delta_i{}^j\, \delta_A{}^B\,, \tag{D.26}$$

and $\mathbb{C}_{AB}$ is the $Sp(n)$-invariant metric. The Vielbein also satisfies the reality condition

$$\left(f_X{}^{iA}\right)^* \;=\; \varepsilon_{ij}\, \mathbb{C}_{AB}\, f_X{}^{jB}\,, \tag{D.27}$$

and they are covariantly constant under the combination of the Levi-Cività, $\mathfrak{su}(2)$- and $\mathfrak{sp}(n)$ connections. The Vielbein also gives us the tangent version of the complex structures. The constant matrices $-i\sigma^r$ satisfy the algebra Eq. (D.20), and we have

$$J^r{}_X{}^Y \;=\; f_X{}^{iA}\, \mathsf{J}^r{}_{iA}{}^{jB}\, f_{jB}{}^Y\,, \qquad \mathsf{J}^r{}_{iA}{}^{jB} \;\equiv\; -i\sigma^r{}_i{}^j\, \delta_A{}^B\,. \tag{D.28}$$

The spin connection can be split into its $\mathfrak{su}(2)$ and $\mathfrak{sp}(n)$ components as follows:

$$\omega_{X\,iA}{}^{jB} \;=\; \tfrac{i}{2}\, \omega_X{}^r\, \mathsf{J}^r{}_{iA}{}^{jB} \;+\; \omega_{X\,A}{}^B\, \delta_i{}^j\,. \tag{D.29}$$

Some useful identities are

$$R_{XY}{}^{r} = \tfrac{1}{4}\nu \, J^{r}{}_{XY} \,, \tag{D.30}$$

$$2 f_{[X}{}^{iA} f_{Y]jA} = i J^{r}{}_{XY} \, \sigma^{r}{}_{j}{}^{i} \,, \tag{D.31}$$

$$2 f_{(X}{}^{iA} f_{Y)jA} = g_{XY} \, \delta_{j}{}^{i} \,. \tag{D.32}$$

The constant ν is given in terms of the dimensionality of the manifold $4n$ and its Ricci scalar R by

$$\nu = \frac{R}{4n(n+2)} \,. \tag{D.33}$$

E

Gauging isometries

In this appendix we are going to review briefly the gauging of the isometries of the scalar manifolds of $N = 1, d = 5$ supergravity in order to clarify some definitions and conventions.

E.1 Killing vectors and gauge transformations

The complete scalar manifold (or target space) of the scalar fields of $N = 1, d = 5$ supergravity is the product of a real special manifold and a quaternionic Kähler manifold parametrized, respectively, by the scalars of the vector supermultiplets (ϕ^x) and by the scalars of the hypermultiplets (q^X). The metrics of these two manifolds are denoted by $g_{xy}(\phi)$ and $g_{XY}(q)$.

We can describe the most general $N = 1, d = 5$ gauged supergravity theory by focusing on the gauging of the isometries of the scalar manifolds. In the end we will see that there are gaugings (necessarily Abelian) unrelated to isometries that fit in the general description.

The isometries to be gauged are generated by Killing vectors of the real special manifold $k_I{}^x(\phi)\partial_x$ and the quaternionic Kähler manifold $k_I{}^X(q)\partial_X$, a pair for each vector $A^I{}_\mu$ of the theory, although some (or all) can be identically zero.

The isometries generated by the Killing vectors $k_I{}^X$ act on the quaternions according to

$$\delta_\Lambda q^X = -g\Lambda^I k_I{}^X \,. \tag{E.1}$$

In the gauged theory the Λ^Is are the local parameters of vector gauge transformations

$$\delta_\Lambda A^I{}_\mu = \partial_\mu \Lambda^I + g f_{JK}{}^I A^J{}_\mu \Lambda^K \,, \tag{E.2}$$

where $f_{JK}{}^I$ are the structure constants of the gauge group G and are given by the Lie brackets of the $k_I{}^X$s

$$[k_I, k_J] = -f_{IJ}{}^K k_K \,. \tag{E.3}$$

This implies that the functions h^I of the real special manifold transform in the adjoint representation of G:

$$\delta_\Lambda h^I = -g f_{JK}{}^I \Lambda^J h^K \,. \tag{E.4}$$

In turn, this implies for the scalars themselves

$$\delta_\Lambda \phi^x = -g \Lambda^I k_I{}^x \,, \tag{E.5}$$

where

$$k_I{}^x = -\sqrt{3} f_{IJ}{}^K h^J h^x_K \,. \tag{E.6}$$

These objects must be Killing vectors of $g_{xy}(\phi)$ if the Λ^I transformations are also symmetries of the corresponding σ model. Writing $g_{xy}\partial\phi^x\partial\phi^y = -2C_{IJKL}h^I\partial h^J\partial h^K$ it is easy to see that necessary and sufficient condition is

$$f_{I(J}{}^K C_{MN)K} = 0 \,, \tag{E.7}$$

i.e. that C_{IJK} is an invariant tensor.

Furthermore, the Killing vectors $k_I{}^x(\phi)$ satisfy the same Lie algebra as the $k_I{}^X(q)$s and can also be written in the equivalent form

$$k_I{}^x = -\sqrt{3} f_{IJ}{}^K h^{Jx} h_K \,. \tag{E.8}$$

The G-covariant derivatives on the scalars are

$$\mathfrak{D}_\mu \phi^x = \partial_\mu \phi^x + g A^I{}_\mu k_I{}^x \,, \qquad \mathfrak{D}_\mu h^I = \partial_\mu h^I + g f_{JK}{}^I A^J{}_\mu h^K \,, \tag{E.9}$$

$$\mathfrak{D}_\mu q^X = \partial_\mu q^X + g A^I{}_\mu k_I{}^X \,, \tag{E.10}$$

and they transform covariantly as

$$\delta_\Lambda \mathfrak{D}_\mu \varphi^{\tilde{x}} = -g\Lambda^I \partial_{\tilde{y}} k_I{}^{\tilde{x}} \mathfrak{D}_\mu \varphi^{\tilde{y}} \,, \qquad \delta_\Lambda \mathfrak{D}_\mu h^I = -g f_{JK}{}^I \Lambda^J \mathfrak{D}_\mu h^K \,, \tag{E.11}$$

where we have unified the notation on the scalars, $\varphi^{\tilde{x}} = (\phi^x, q^X)$, $k_I{}^{\tilde{x}} = (k_I{}^x, k_I{}^X)$.

The second derivatives are defined by

$$\mathfrak{D}_\mu \mathfrak{D}_\nu \varphi^{\tilde{x}} \equiv \nabla_\mu \mathfrak{D}_\nu \varphi^{\tilde{x}} + \Gamma_{\tilde{y}\tilde{z}}{}^{\tilde{x}} \mathfrak{D}_\mu \varphi^{\tilde{y}} \mathfrak{D}_\mu \varphi^{\tilde{z}} + g A^I{}_\mu \partial_{\tilde{y}} k_I{}^{\tilde{x}} \mathfrak{D}_\nu \varphi^{\tilde{y}} \,, \tag{E.12}$$

where $\Gamma_{\tilde{y}\tilde{z}}{}^{\tilde{x}}$ are the target space Christoffel symbols. Their transformations and commutator are given by

$$\delta_\Lambda \mathfrak{D}_\mu \mathfrak{D}_\nu \varphi^{\tilde{x}} = -g\Lambda^I \partial_{\tilde{y}} k_I{}^{\tilde{x}} \mathfrak{D}_\mu \mathfrak{D}_\nu \varphi^{\tilde{y}} \,, \tag{E.13}$$

$$[\mathfrak{D}_\mu, \mathfrak{D}_\nu]\, \varphi^{\tilde{x}} = g F^I{}_{\mu\nu} k_I{}^{\tilde{x}} \,, \tag{E.14}$$

where $F^I{}_{\mu\nu}$ is the gauge field strength

$$F^I{}_{\mu\nu} = 2\partial_{[\mu} A^I{}_{\nu]} + g f_{JK}{}^I A^J{}_\mu A^K{}_\nu \,. \tag{E.15}$$

All these definitions are enough to construct a gauge-invariant action for the scalars, since this essentially depends on the target space metric. However, they are not enough to gauge the full supergravity theory, which depends on other structures as well. In particular, it depends on the complex structures of the hyperscalar manifold and we have to study under which conditions they are preserved by the gauging.

E.2 The covariant Lie derivative and the momentum map

This appendix concerns only to the hyperscalar sector of the target manifold. The quaternionic Kähler geometry of this manifold is defined not only by the metric g_{XY} but by the quaternionic structure $\vec{J}_X{}^Y$, which should also be preserved by the symmetries to be gauged. Therefore, one must require the vanishing of the Lie derivative of the quaternionic structure with respect to the Killing vectors $k_I{}^X$. One has to use an $SU(2)$-*covariant* Lie derivative for consistency or, as it is usually done in the literature, impose the vanishing of the standard Lie derivative up to gauge transformations. Here we will use an $SU(2)$-covariant Lie derivative whose construction we describe first.

Let $\vec{\psi}$ by an $SU(2)$ vector and, simultenously an arbitrary tensor on the hyperscalar variety, and $\vec{\omega}$ the $SU(2)$ connection. Under infinitesimal $SU(2)$ gauge transformations

$$\delta_\lambda \vec{\psi} = -2\vec{\lambda}(q) \times \vec{\psi}\,, \qquad \delta_\lambda \vec{\omega} = -2\vec{\lambda}(q) \times \vec{\omega} + d\vec{\lambda}(q)\,. \tag{E.16}$$

The standard Lie derivative of $\vec{\psi}$ along the vector ${k_I}^X$ (denoted by $\mathcal{L}_I\vec{\psi}$) transforms under $SU(2)$ as

$$\delta_\lambda \mathcal{L}_I \vec{\psi} = -2\vec{\lambda} \times \mathcal{L}_I\vec{\psi} - 2\partial_I\vec{\lambda} \times \vec{\psi}\,, \tag{E.17}$$

where $\partial_I \equiv {k_I}^X\partial_X$. We now want to find another definition of Lie derivative that transforms without derivatives of the transformation parameter. Introducing for each Killing vector[1] ${k_I}^X$ a $\vec{\eta}_I$ transforming as

$$\delta_\lambda \vec{\eta}_I = -2\vec{\lambda} \times \vec{\eta}_I + \partial_I\vec{\lambda}\,, \tag{E.18}$$

we define the $SU(2)$-covariant Lie derivative on $SU(2)$ vectors

$$\mathbb{L}_I\vec{\psi} \equiv \mathcal{L}_I\vec{\psi} + 2\vec{\eta}_I \times \vec{\psi}\,. \tag{E.19}$$

For this to be a good definition $\mathbb{L}_I$ must satisfy the standard properties of a Lie derivative.

$\mathbb{L}_I$ is clearly a linear operator and it satisfies the Leibnitz rule for products of $SU(2)$ vectors such as $\vec{\psi}\cdot\vec{\phi}$ and $\vec{\psi}\times\vec{\phi}$. The Lie derivative must also satisfy

$$[\mathbb{L}_I, \mathbb{L}_J] = \mathbb{L}_{[k_I,k_J]}\,, \tag{E.20}$$

which implies the Jacobi identity. This requires the "curvature" of the "connection" $\vec{\eta}_I$ to be

$$\partial_I\vec{\eta}_J - \partial_J\vec{\eta}_I + 2\vec{\eta}_I \times \vec{\eta}_J = -{f_{IJ}}^K\vec{\eta}_K\,. \tag{E.21}$$

It should be clear that $\vec{\eta}_I$ must be related with the $SU(2)$ connection $\vec{\omega}$, but it is not just ${k_I}^X\vec{\omega}_X$, which has the right transformation property Eq. (E.18) but does not satisfy curvature property Eq. (E.21). Thus, we introduce yet another $SU(2)$ vector[2]

$$\vec{\eta}_I = {k_I}^X\vec{\omega}_X - \tfrac{1}{2}\vec{P}_I\,, \tag{E.22}$$

[1]Only covariant Lie derivatives with respect to Killing vectors can be properly defined.

[2]We put the $-1/2$ factor to agree with the conventions of Ref. [65]

which must satisfy

$$\mathfrak{D}_I \vec{P}_J - \mathfrak{D}_J \vec{P}_I - \vec{P}_I \times \vec{P}_J + \tfrac{1}{2} k_I{}^X \vec{J}_{XY} k_J{}^Y = f_{IJ}{}^K \vec{P}_K \,, \tag{E.23}$$

in order to meet Eq. (E.21). Here we have used the fact that in quaternionic Kähler manifolds the curvature of the $SU(2)$ connection is non-vanishing and proportional to the Kähler two-forms. We are going to show that $\vec{P}_I$ satisfies the equation that defines it as a *momentum map*.

Now, assuming that a $\vec{P}_I$ satisfying Eq. (E.23) has been found, we can write the conditions that the vector $k_I{}^X$ must satisfy to be the generator of a symmetry of the hyperscalar manifold in the form

$$\mathbb{L}_I g_{XY} = 0 \,, \tag{E.24}$$

$$\mathbb{L}_I \vec{J}_{XY} = 0 \,. \tag{E.25}$$

The first equation is just the Killing equation since $\mathbb{L}_I g_{XY} = \mathcal{L}_I g_{XY}$. Given the metric and quaternionic structure, the second condition (*triholomorphicity* of the Killing vectors) can be seen as a condition for $\vec{P}_I$ just as the Killing equation can be seen as a condition for k_I once the metric g_{XY} is given: it can be written in the form

$$-\vec{J}_X{}^Y \times \vec{P}_I = \nabla_X k_I{}^Z \vec{J}_Z{}^Y - \vec{J}_X{}^Z \nabla_Z k_I{}^Y \,, \tag{E.26}$$

which says that $\vec{P}_I$ measures the commutator between the quaternionic structure and the covariant derivative of the Killing vectors. By contracting this equation with $\vec{J}_Y{}^X$ we obtain an expression for $\vec{P}_I$ itself, valid for $n_H \neq 0$[3]

$$2n_H \vec{P}_I = \vec{J}_X{}^Y \nabla_Y k_I{}^X \,. \tag{E.30}$$

[3] In absence of hypermultiplets ($n_H = 0$) the momentum map $\vec{P}_I$ can still be defined in two cases in which they are equivalent to a set of constant Fayet-Iliopoulos terms. In the first case the gauge group contains an $SU(2)$ factor and

$$\vec{P}_I = \vec{e}_I \, \xi \,, \tag{E.27}$$

where ξ is an arbitrary constant and the $\vec{e}_I$ are constants that are nonzero for I in the range of the $SU(2)$ factor and satisfy

$$\vec{e}_I \times \vec{e}_J = f_{IJ}{}^K \vec{e}_K \,. \tag{E.28}$$

In the second case the gauge group contains a $U(1)$ factor and

$$\vec{P}_I = \vec{e} \, \xi_I \,, \tag{E.29}$$

where $\vec{e}$ is an arbitrary $SU(2)$ vector and the ξ_Is are arbitrary constants that are nonzero for I corresponding to the $U(1)$ factor.

For this solution to be consistent, it has to satisfy Eq. (E.23). To see it we first take the derivative of the above solution Eq. (E.30) using the following identity for Killing vectors,

$$\nabla_X \nabla_Y k^Z = R_{XWY}{}^Z k^W \,, \tag{E.31}$$

and the canonical decomposition of the curvature between its $SU(2)$ and $Sp(n_H)$ parts,

$$R_{XWY}{}^Z = -\vec{J}_Y{}^Z \cdot \vec{\mathcal{R}}_{XW} + f_Y{}^{iB} f_{iA}{}^Z \mathcal{R}_{XW\,B}{}^A \,. \tag{E.32}$$

Only the $SU(2)$ part of the curvature contributes to the derivative of $\vec{P}_I$:

$$\mathfrak{D}_X \vec{P}_I = 2\vec{\mathcal{R}}_{XY} k_I{}^Y = -\tfrac{1}{2}\vec{J}_{XY} k_I{}^Y \,. \tag{E.33}$$

This equation can alternatively be taken as the definition of $\vec{P}_I$. It defines a momentum map and it is crucial for coupling hypermultiplets to supergravity. Observe that the integrability condition of Eq. (E.33) is precisely Eq. (E.26).

We can now substitute Eq. (E.33) in Eq. (E.23), obtaining

$$\vec{P}_I \times \vec{P}_J + \tfrac{1}{2} k_I{}^X \vec{J}_{XY} k_J{}^Y = f_{IJ}{}^K \vec{P}_K \,. \tag{E.34}$$

On the other hand, contracting Eq. (E.26) with $\nabla_Y k_J{}^X$ we get

$$n_H \vec{P}_I \times \vec{P}_J = -\vec{J}_X{}^Y \nabla_Y k_{[I|}{}^Z \nabla_Z k_{|J]}{}^X \,, \tag{E.35}$$

integrating by parts the right hand side of this expression, using the algebra of the Killing vectors, identity (E.31), the Bianchi identity of the curvature and the curvature decomposition (E.32) one recovers Eq. (E.34).

From Eq. (E.30) one can see that the momentum map is also covariantly preserved by the Killing vectors

$$\mathbb{L}_I \vec{P}_J = 0 \,. \tag{E.36}$$

There is still one more consistency check on the momentum map: the quaternionic Kähler two-form is $SU(2)$-covariantly closed. To ensure that this property is consistent with Eq. (E.25) we must check that the covariant Lie derivative commutes with the $SU(2)$-covariant exterior derivative, in analogy to the commutation between standard Lie derivatives and exterior derivatives. This requirement leads us to the condition

$$\mathcal{L}_I \vec{\omega} - d\vec{\eta}_I - 2\vec{\omega} \times \vec{\eta}_I = 0 \,. \tag{E.37}$$

Notice that this relation between the two $SU(2)$ connections is in principle independent of Eq. (E.22). After substitution of Eq. (E.22) in Eq. (E.37) the latter becomes the differential definition of $\vec{P}_I$, Eq. (E.33).

Eq. (E.33) can alternatively be used to solve the Killing vectors in terms of the derivatives of the momentum map,

$$k_I{}^X = \tfrac{2}{3}\vec{J}^{XY} \cdot \mathfrak{D}_Y \vec{P}_I \,. \tag{E.38}$$

In view of this relation $\vec{P}_I$ is sometimes called the prepotential.

The moment map assigns a triplet of real numbers to each Killing vector. The Killing vectors realize the algebra of G. Eq. (E.34) can also be understood as a realization of the algebra of G in terms of $\vec{P}_I$, $\vec{J}_{XY}$ being the symplectic structure used to define the Poisson brackets which are the left hand side of Eq. (E.34).

In summary, given a Killing vector of the metric $g_{XY}(q)$ we can always construct the momentum map $\vec{P}_I$ by Eq. (E.30). Next we define the covariant Lie derivative along the Killing vector by means of the connection $\vec{\eta}_I$. This covariant Lie derivative enjoys the algebraic and differential properties of a pure Lie derivative and also commutes with covariant exterior derivatives. The Killing vector becomes automatically covariantly triholomorphic according to Eq. (E.25).

E.3 $SU(2)$ transformations induced by G

Let us now consider the momentum map as a composite spacetime field over which depends only on the q^Xs. Under general variations δq^X and using the definition of the momentum map (E.33),

$$\delta\vec{P}_I = -\delta q^X \left(\tfrac{1}{2}\vec{J}_{XY} k_I{}^Y + 2\vec{\omega}_X \times \vec{P}_I\right) \,. \tag{E.39}$$

If this transformation is a G-gauge transformation $\delta_\Lambda q^X = -g\Lambda^J k_J{}^X$, taking into account Eq. (E.34), we obtain

$$\delta_\Lambda \vec{P}_I = -g f_{IJ}{}^K \Lambda^J \vec{P}_K + 2g\Lambda^J \vec{\eta}_J \times \vec{P}_I \,, \tag{E.40}$$

which is the adjoint action of G on $\vec{P}_I$ plus an induced $SU(2)$ gauge transformation with parameter $-g\Lambda^J\vec{\eta}_J$ which is present even if G is Abelian. This is the mechanism through which G can act on objects such as the spinors of the supergravity theory which only have $SU(2)$ indices, opening the doors to the gauging of groups larger than

$SU(2)$: if the gravitino transforms under standard $SU(2)$ transformations according to

$$\delta_\lambda \psi^i_\mu = i\psi^j_\mu \vec{\sigma}_j{}^i \cdot \vec{\lambda}\,, \tag{E.41}$$

where $\vec{\lambda}$ is the infinitesimal $SU(2)$ parameter, then, under G-gauge transformations it will undergo a similar transformation with $\vec{\lambda} = -g\Lambda^I \vec{\eta}_I$.

Thus, in G-gauged supergravity the pullback of the $SU(2)$ connection that couples to the spinors of the theory has to be replaced by

$$\vec{B} \equiv \vec{A} + \tfrac{1}{2} g A^I \vec{P}_I\,, \qquad \vec{A} \equiv dq^X \vec{\omega}_X\,, \tag{E.42}$$

to take into account the $SU(2)$ transformations induced by G-gauge transformations, which act on it as

$$\delta_\Lambda \vec{B} = -2(-g\Lambda^I \vec{\eta}) \times \vec{B} + d(-g\Lambda^I \vec{\eta})\,. \tag{E.43}$$

The covariant derivative on these objects is

$$\mathfrak{D}_\mu \psi^i_\nu = \nabla_\mu \psi^i_\nu + \psi^j B_{\mu j}{}^i\,. \tag{E.44}$$

F

Proofs of some identities

Let us consider the generalized stabilization equations derived from Eq. (4.3). Differentiating the imaginary part of that equation (i.e. Eq.(4.4)), we get

$$d\mathcal{I}_\Lambda = d\Im\mathrm{m}\mathcal{F}_\Lambda = \tfrac{1}{2i}(d\mathcal{X}^\Lambda \mathcal{F}_{\Sigma\Lambda} - d\mathcal{X}^{*\,\Lambda}\mathcal{F}^*_{\Sigma\Lambda}) = d\mathcal{R}^\Sigma \Im\mathrm{m}\mathcal{F}_{\Sigma\Lambda} + d\mathcal{I}^\Sigma \Re\mathrm{e}\mathcal{F}_{\Sigma\Lambda}\,, \tag{F.1}$$

where we have used $\mathcal{X}^\Lambda = \mathcal{R}^\Lambda + i\mathcal{I}^\Lambda$. Using the invertibility of the imaginary part of $\mathcal{F}_{\Sigma\Lambda}$ we get

$$d\mathcal{R}^\Sigma = \Im\mathrm{m}\mathcal{F}^{\Sigma\Lambda} d\mathcal{I}_\Lambda - \Im\mathrm{m}\mathcal{F}^{\Sigma\Omega}\Re\mathrm{e}\mathcal{F}_{\Omega\Lambda} d\mathcal{I}^\Lambda\,. \tag{F.2}$$

On the other hand, differentiating the real part of Eq. (4.3)

$$d\mathcal{R}_\Lambda = d\Re\mathrm{e}\mathcal{F}_\Lambda = \tfrac{1}{2}(d\mathcal{X}^\Lambda \mathcal{F}_{\Sigma\Lambda} + d\mathcal{X}^{*\,\Lambda}\mathcal{F}^*_{\Sigma\Lambda}) = d\mathcal{R}^\Sigma \Re\mathrm{e}\mathcal{F}_{\Sigma\Lambda} - d\mathcal{I}^\Sigma \Im\mathrm{m}\mathcal{F}_{\Sigma\Lambda}\,, \tag{F.3}$$

and, substituting our previous result for $d\mathcal{R}^\Lambda$

$$d\mathcal{R}_\Sigma = \Re\mathrm{e}\mathcal{F}_{\Sigma\Omega}\Im\mathrm{m}\mathcal{F}^{\Omega\Lambda} dH_\Lambda - (\Im\mathrm{m}\mathcal{F}_{\Sigma\Lambda} + \Re\mathrm{e}\mathcal{F}_{\Sigma\Omega}\Im\mathrm{m}\mathcal{F}^{\Omega\Delta}\Re\mathrm{e}\mathcal{F}_{\Delta\Lambda})d\mathcal{I}^\Lambda\,. \tag{F.4}$$

We can write all these results in the form

$$d\mathcal{R} = \begin{pmatrix} -\Im\mathrm{m}\mathcal{F}^{-1}\Re\mathrm{e}\mathcal{F} & \Im\mathrm{m}\mathcal{F}^{-1} \\ \\ -(\Im\mathrm{m}\mathcal{F} + \Re\mathrm{e}\mathcal{F}\Im\mathrm{m}\mathcal{F}^{-1}\Re\mathrm{e}\mathcal{F}) & \Re\mathrm{e}\mathcal{F}\Im\mathrm{m}\mathcal{F}^{-1} \end{pmatrix} d\mathcal{I}\,, \tag{F.5}$$

$$d\mathcal{I} = \begin{pmatrix} \Im\mathrm{m}\mathcal{F}^{-1}\Re\mathrm{e}\mathcal{F} & -\Im\mathrm{m}\mathcal{F}^{-1} \\ \\ \Im\mathrm{m}\mathcal{F} + \Re\mathrm{e}\mathcal{F}\Im\mathrm{m}\mathcal{F}^{-1}\Re\mathrm{e}\mathcal{F} & -\Re\mathrm{e}\mathcal{F}\Im\mathrm{m}\mathcal{F}^{-1} \end{pmatrix} d\mathcal{R}\,, \tag{F.6}$$

from which we can read identities such as

$$
\begin{aligned}
\frac{\partial\mathcal{R}^\Sigma}{\partial\mathcal{I}_\Lambda} &= \frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}_\Sigma} = \frac{\partial\mathcal{I}^\Lambda}{\partial\mathcal{R}_\Sigma}\,, &\qquad \frac{\partial\mathcal{R}_\Sigma}{\partial\mathcal{I}^\Lambda} &= \frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}^\Sigma} = -\frac{\partial\mathcal{I}_\Lambda}{\partial\mathcal{R}^\Sigma}\,,\\
\frac{\partial\mathcal{R}^\Sigma}{\partial\mathcal{I}^\Lambda} &= -\frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}_\Sigma} = \frac{\partial\mathcal{I}_\Lambda}{\partial\mathcal{R}_\Sigma}\,, &\qquad \frac{\partial\mathcal{R}_\Sigma}{\partial\mathcal{I}_\Lambda} &= -\frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}^\Sigma} = \frac{\partial\mathcal{I}^\Lambda}{\partial\mathcal{R}^\Sigma}\,.
\end{aligned}
\tag{F.7}
$$

We can now prove Eq. (4.46): taking the derivative of $\mathcal{R}$ as a function of $\mathcal{I}$ we have

$$
\begin{aligned}
\langle\, \nabla_\mu\mathcal{R} \mid \mathcal{I}\,\rangle &= \langle\, \frac{\partial\mathcal{R}}{\partial\mathcal{I}^\Lambda}\nabla_\mu\mathcal{I}^\Lambda + \frac{\partial\mathcal{R}}{\partial\mathcal{I}_\Lambda}\nabla_\mu\mathcal{I}_\Lambda \mid \mathcal{I}\,\rangle\\
&= \nabla_\mu\mathcal{I}^\Lambda\left(\mathcal{I}^\Sigma\frac{\partial\mathcal{R}_\Sigma}{\partial\mathcal{I}^\Lambda} - \mathcal{I}_\Sigma\frac{\partial\mathcal{R}^\Sigma}{\partial\mathcal{I}^\Lambda}\right) + \nabla_\mu\mathcal{I}_\Lambda\left(\mathcal{I}^\Sigma\frac{\partial\mathcal{R}_\Sigma}{\partial\mathcal{I}_\Lambda} - \mathcal{I}_\Sigma\frac{\partial\mathcal{R}^\Sigma}{\partial\mathcal{I}_\Lambda}\right)\,,
\end{aligned}
\tag{F.8}
$$

and using now the above relations between partial derivatives

$$
\langle\, \nabla_\mu\mathcal{R} \mid \mathcal{I}\,\rangle = \nabla_\mu\mathcal{I}^\Lambda\left(\mathcal{I}^\Sigma\frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}^\Sigma} + \mathcal{I}_\Sigma\frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}_\Sigma}\right) - \nabla_\mu\mathcal{I}_\Lambda\left(\mathcal{I}^\Sigma\frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}^\Sigma} + \mathcal{I}_\Sigma\frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}_\Sigma}\right)\,. \tag{F.9}
$$

Given that the real section $\mathcal{R}$ is homogeneous of first order in the $\mathcal{I}$'s

$$
\mathcal{I}^\Sigma\frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}^\Sigma} + \mathcal{I}_\Sigma\frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}_\Sigma} = \mathcal{R}_\Lambda\,, \qquad \mathcal{I}^\Sigma\frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}^\Sigma} + \mathcal{I}_\Sigma\frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}_\Sigma} = \mathcal{R}^\Lambda\,, \tag{F.10}
$$

which proves the identity.

Similarly, expanding the r.h.s. of Eq. (4.45) we get

$$
\langle\, \mathcal{R} \mid \nabla_\mu\mathcal{R}\,\rangle = \left(\frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}^\Sigma}R_\Lambda - \frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}^\Sigma}R^\Lambda\right)d\mathcal{I}^\Sigma + \left(\frac{\partial\mathcal{R}^\Lambda}{\partial\mathcal{I}_\Sigma}R_\Lambda - \frac{\partial\mathcal{R}_\Lambda}{\partial\mathcal{I}_\Sigma}R^\Lambda\right)d\mathcal{I}_\Sigma\,, \tag{F.11}
$$

and using the identities between partial derivatives and the fact that the real section $\mathcal{I}$ is homogeneous of first order in $\mathcal{R}$, we arrive at the result we wanted.

Bibliography

[1] Sergio L. Cacciatori, Alessio Celi, and Daniela Zanon. BPS equations in $N = 2$, $D = 5$ supergravity with hypermultiplets. *Class. Quant. Grav.*, 20:1503–1518, 2003.

[2] Sergio L. Cacciatori and Alessio Celi. New constraint for black holes in $N = 2$, $D = 5$ supergravity with matter. hep-th/0405284, 2004.

[3] Patrick Meessen and Tomás Ortín. The supersymmetric configurations of $N = 2$, $d = 4$ supergravity coupled to vector supermultiplets. *Nucl. Phys.*, B749:291–324, 2006.

[4] Mechthild Huebscher, Patrick Meessen, and Tomás Ortín. Supersymmetric solutions of $N = 2$ $d = 4$ SUGRA: The whole ungauged shebang. *Nucl. Phys.*, B759:228–248, 2006.

[5] K. P. Tod. More on supercovariantly constant spinors. *Class. Quant. Grav.*, 12:1801, 1995.

[6] R. Kallosh and T. Ortín. Killing spinor identities. hep-th/9306085, 1993.

[7] Mirjam Cvetic and Donam Youm. Dyonic BPS saturated black holes of heterotic string on a six torus. *Phys. Rev.*, D53:584–588, 1996.

[8] Mirjam Cvetic and Arkady A. Tseytlin. Solitonic strings and BPS saturated dyonic black holes. *Phys. Rev.*, D53:5619–5633, 1996.

[9] Mirjam Cvetic and Christopher M. Hull. Black holes and U-duality. *Nucl. Phys.*, B480:296–316, 1996.

[10] Sergio Ferrara and Renata Kallosh. Universality of Supersymmetric Attractors. *Phys. Rev.*, D54:1525–1534, 1996.

[11] K. P. Tod. All metrics admitting supercovariantly constant spinors. *Phys. Lett.*, B121:241, 1983.

[12] G. W. Gibbons and C. M. Hull. A Bogomol'nyi bound for General Relativity and solitons in $N = 2$ Supergravity. *Phys. Lett.*, B109:190, 1982.

[13] E. Bergshoeff, R. Kallosh, and T. Ortín. Stationary axion/dilaton solutions and supersymmetry. *Nucl. Phys.*, B478:156, 1996.

[14] Z. Perjés. Solutions of the coupled Einstein-Maxwell equations representing the fields of spinning sources. *Phys. Rev. Lett.*, 27:1668, 1971.

[15] W. Israel and G. A. Wilson. A class of stationary electromagnetic vacuum fields. *J. Math. Phys.*, 13:865, 1972.

[16] G. W. Gibbons. Antigravitating black hole solitons with scalar hair in $N = 4$ Supergravity. *Nucl. Phys.*, B207:337, 1982.

[17] G. W. Gibbons and K. Maeda. Black holes and membranes in higher dimensional theories with dilaton fields. *Nucl. Phys.*, B298:741, 1988.

[18] D. Garfinkle, G. Horowitz, and A. Strominger. Charged black holes in String Theory. *Phys. Rev.*, D43:3140, 1991. [D. Garfinkle, G. Horowitz, and A. Strominger, *Phys. Rev.*, D45:3888, 1992. Erratum].

[19] A. Shapere, S. Trivedi, and F. Wilczek. Dual dilaton dyons. *Mod. Phys. Lett.*, A6:2677, 1991.

[20] R. Kallosh, A. Linde, T. Ortín, A. Peet, and A. Van Proeyen. Supersymmetry as a cosmic censor. *Phys. Rev.*, D46:5278, 1992.

[21] T. Ortín. Electric - magnetic duality and supersymmetry in stringy black holes. *Phys. Rev.*, D47:3136, 1993.

[22] R. Kallosh and T. Ortín. Charge quantization of axion - dilaton black holes. *Phys. Rev.*, D48:742, 1993.

[23] T. Ortín. $SL(2,\mathbb{R})$ duality covariance of Killing spinors in axion - dilaton black holes. *Phys. Rev.*, D51:790, 1995.

[24] R. Kallosh, D. Kastor, T. Ortín, and T. Torma. Supersymmetry and stationary solutions in dilaton axion gravity. *Phys. Rev.*, D50:6374, 1994.

[25] M. Rogatko. Stationary axisymmetric axion - dilaton black holes: Mass formulae. *Class. Quant. Grav.*, 11:689, 1994.

[26] D. Gal'tsov and O. Kechkin. Ehlers-Harrison type transformations in dilaton - axion gravity. *Phys. Rev.*, D50:7394, 1994.

[27] D. Gal'tsov and O. Kechkin. U duality and symplectic formulation of dilaton - axion gravity. *Phys. Rev.*, D54:1656, 1996.

[28] D. Gal'tsov and O. Kechkin. Matrix dilaton - axion for heterotic string in three-dimensions. *Phys. Lett.*, B361:52, 1995.

[29] A. García, D. Gal'tsov, and O. Kechkin. Class of stationary axisymmetric solutions of the Einstein-Maxwell dilaton - axion field equations. *Phys. Rev. Lett.*, 74:1276, 1995.

[30] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos. A new maximally supersymmetric background of IIB Superstring Theory. *JHEP*, 01:047, 2002.

[31] J. Kowalski-Glikman. Vacuum states in supersymmetric Kaluza-Klein Theory. *Phys. Lett.*, B134:194, 1984.

[32] J. Kowalski-Glikman. Positive energy theorem and vacuum states for the Einstein-Maxwell system. *Phys. Lett.*, B150:125, 1985.

[33] P. Meessen. A small note on *pp*-wave vacua in 6 and 5 dimensions. *Phys. Rev.*, D65:087501, 2002.

[34] J. Figueroa-O'Farrill and G. Papadopoulos. Maximally supersymmetric solutions of ten- and eleven- dimensional supergravities. *JHEP*, 03:048, 2003.

[35] J. P. Gauntlett, J. B. Gutowski, C. Hull, S. Pakis, and H. S. Reall. All supersymmetric solutions of minimal supergravity in five dimensions. *Class. Quant. Grav.*, 20:4587, 2003.

[36] J. P. Gauntlett and J. B. Gutowski. All supersymmetric solutions of minimal gauged supergravity in five dimensions. *Phys. Rev.*, D68:105009, 2003. [J. P. Gauntlett and J. B. Gutowski, *Phys. Rev.* D70:089901, 2004. Erratum].

[37] M. Caldarelli and D. Klemm. All supersymmetric solutions of $N = 2$, $D = 4$ gauged supergravity. *JHEP*, 09:019, 2003.

[38] J. B. Gutowski, D. Martelli, and H. S. Reall. All supersymmetric solutions of minimal supergravity in six dimensions. *Class. Quant. Grav.*, 20:5049, 2003.

[39] A. Chamseddine, J. Figueroa-O'Farrill, and W. Sabra. Supergravity vacua and Lorentzian Lie groups. hep-th/0306278, 2003.

[40] Alessio Celi. Toward the classification of BPS solutions of $N = 2$, $d = 5$ gauged supergravity with matter couplings. hep-th/0405283, 2003.

[41] D. C. Jong, A. Kaya, and E. Sezgin. 6D dyonic string with active hyperscalars. *JHEP*, 11:047, 2006.

[42] Iosif Bena and Per Kraus. Microscopic description of black rings in AdS/CFT. *JHEP*, 12:070, 2004.

[43] Iosif Bena and Per Kraus. Microstates of the D1-D5-KK system. *Phys. Rev.*, D72:025007, 2005.

[44] Davide Gaiotto, Andrew Strominger, and Xi Yin. New connections between 4D and 5D black holes. *JHEP*, 02:024, 2006.

[45] Henriette Elvang, Roberto Emparan, David Mateos, and Harvey S. Reall. Supersymmetric 4D rotating black holes from 5D black rings. *JHEP*, 08:042, 2005.

[46] Davide Gaiotto, Andrew Strominger, and Xi Yin. 5D black rings and 4D black holes. *JHEP*, 02:023, 2006.

[47] Iosif Bena, Per Kraus, and Nicholas P. Warner. Black rings in Taub-NUT. *Phys. Rev.*, D72:084019, 2005.

[48] Klaus Behrndt, Gabriel Lopes Cardoso, and Swapna Mahapatra. Exploring the relation between 4D and 5D BPS solutions. *Nucl. Phys.*, B732:200–223, 2006.

[49] R. D. Sorkin. Kaluza-Klein Monopole. *Phys. Rev. Lett.*, 51:87, 1983.

[50] D.J. Gross and M. J. Perry. Magnetic monopoles in Kaluza-Klein theories. *Nucl. Phys.*, B226:29, 1983.

[51] Mirjam Cvetic and Finn Larsen. Near horizon geometry of rotating black holes in five dimensions. *Nucl. Phys.*, B531:239–255, 1998.

[52] Oleg Lunin and Samir D. Mathur. AdS/CFT duality and the black hole information paradox. *Nucl. Phys.*, B623:342–394, 2002.

[53] Samir D. Mathur. The fuzzball proposal for black holes: An elementary review. *Fortsch. Phys.*, 53:793–827, 2005.

[54] Oleg Lunin and Samir D. Mathur. Statistical interpretation of Bekenstein entropy for systems with a stretched horizon. *Phys. Rev. Lett.*, 88:211303, 2002.

[55] Oleg Lunin, Samir D. Mathur, and Ashish Saxena. What is the gravity dual of a chiral primary? *Nucl. Phys.*, B655:185–217, 2003.

[56] Samir D. Mathur, Ashish Saxena, and Yogesh K. Srivastava. Constructing 'hair' for the three charge hole. *Nucl. Phys.*, B680:415–449, 2004.

[57] Oleg Lunin. Adding momentum to D1-D5 system. *JHEP*, 04:054, 2004.

[58] Stefano Giusto, Samir D. Mathur, and Ashish Saxena. Dual geometries for a set of 3-charge microstates. *Nucl. Phys.*, B701:357–379, 2004.

[59] Stefano Giusto, Samir D. Mathur, and Ashish Saxena. 3-charge geometries and their CFT duals. *Nucl. Phys.*, B710:425–463, 2005.

[60] Stefano Giusto and Samir D. Mathur. Geometry of D1-D5-P bound states. *Nucl. Phys.*, B729:203–220, 2005.

[61] Henriette Elvang, Roberto Emparan, David Mateos, and Harvey S. Reall. A supersymmetric black ring. *Phys. Rev. Lett.*, 93:211302, 2004.

[62] Jerome P. Gauntlett and Jan B. Gutowski. Concentric black rings. *Phys. Rev.*, D71:025013, 2005.

[63] Jerome P. Gauntlett and Jan B. Gutowski. General concentric black rings. *Phys. Rev.*, D71:045002, 2005.

[64] G. W. Gibbons and S. W. Hawking. Gravitational multi - instantons. *Phys. Lett.*, B78:430, 1978.

[65] Eric Bergshoeff et al. $N = 2$ supergravity in five dimensions revisited. *Class. Quant. Grav.*, 21:3015–3042, 2004.

[66] Ernesto Lozano-Tellechea, Patrick Meessen, and Tomás Ortín. On $d = 4, 5, 6$ vacua with 8 supercharges. *Class. Quant. Grav.*, 19:5921–5934, 2002.

[67] Patrick Meessen and Tomás Ortín. Gödel spacetimes, Abelian instantons, the graviphoton background and other flacuum solutions. *Nucl. Phys.*, B684:235–263, 2004.

[68] M. Günaydin, G. Sierra, and P. K. Townsend. The Geometry Of N=2 Maxwell-Einstein Supergravity And Jordan Algebras. *Nucl. Phys.*, B242:244, 1984.

[69] M. Günaydin, G. Sierra, and P. K. Townsend. Gauging The $D = 5$ Maxwell-Einstein Supergravity Theories: More On Jordan Algebras. *Nucl. Phys.*, B253:573, 1985.

[70] A. C. Cadavid, Anna Ceresole, R. D'Auria, and S. Ferrara. Eleven-dimensional supergravity compactified on Calabi-Yau threefolds. *Phys. Lett.*, B357:76–80, 1995.

[71] J. Bellorín and T. Ortín. A note on simple applications of the Killing spinor identities. *Phys. Lett.*, B616:118, 2005.

[72] Jan B. Gutowski and Harvey S. Reall. General supersymmetric $AdS(5)$ black holes. *JHEP*, 04:048, 2004.

[73] J. Chen and J. Li. Quaternionic maps between hyperKähler manifolds. *J. Diff. Geom.*, 55:355, 2000.

[74] Jorge Bellorín, Patrick Meessen, and Tomás Ortín. Supersymmetry, attractors and cosmic censorship. *Nucl. Phys.*, B762:229–255, 2007.

[75] Frederik Denef. Supergravity flows and D-brane stability. *JHEP*, 08:050, 2000.

[76] Brandon Bates and Frederik Denef. Exact solutions for supersymmetric stationary black hole composites. hep-th/0304094, 2003.

[77] R. C. Myers and M. J. Perry. Black Holes In Higher Dimensional Space-Times. *Annals Phys.*, 172:304, 1986.

[78] M. Abramowitz and I. A. Stegun. *Pocketbook of mathematical functions.* Verlag Harri Deutsch, 1984.

[79] G. W. Gibbons and P. J. Ruback. The Hidden Symmetries Of Multicenter Metrics. *Commun. Math. Phys.*, 115:267, 1988.

[80] Jan B. Gutowski and Wafic Sabra. General supersymmetric solutions of five-dimensional supergravity. *JHEP*, 10:039, 2005.

[81] Ali H. Chamseddine and W. A. Sabra. Metrics admitting Killing spinors in five dimensions. *Phys. Lett.*, B426:36–42, 1998.

[82] Ali H. Chamseddine and W. A. Sabra. Magnetic strings in five dimensional gauged supergravity theories. *Phys. Lett.*, B477:329–334, 2000.

[83] Ali H. Chamseddine and W. A. Sabra. Calabi-Yau black holes and enhancement of supersymmetry in five dimensions. *Phys. Lett.*, B460:63–70, 1999.

[84] Marco M. Caldarelli, Dietmar Klemm, and Emanuele Zorzan. Supersymmetric gyratons in five dimensions. hep-th/0610126, 2006.

[85] Jorge Bellorín, Patrick Meessen, and Tomás Ortín. All the supersymmetric solutions of $N = 1$, $d = 5$ ungauged supergravity. *JHEP*, 01:020, 2007.

[86] A. Chamseddine. $N = 4$ Supergravity coupled to $N = 4$ matter. *Nucl. Phys.*, B185:403, 1981.

[87] A. Das. $SO(4)$ Invariant Extended Supergravity. *Phys. Rev.*, D15:2805, 1977.

[88] E. Cremmer, J. Scherk, and S. Ferrara. $SU(4)$ invariant supergravity theory. *Phys. Lett.*, B74:61, 1978.

[89] E. Cremmer, J. Scherk, and S. Ferrara. $U(N)$ Invariance In Extended Supergravity. *Phys. Lett.*, B68:234, 1977.

[90] A. Font, L. E. Ibañez, D. Lüst, and F. Quevedo. Strong-weak coupling duality and nonperturbative effects in String Theory. *Phys. Lett.*, B249:35, 1990.

[91] C. M. Hull and P. K. Townsend. Unity of superstring dualities. *Nucl. Phys.*, B438:109, 1995.

[92] Brian R. Greene, Alfred D. Shapere, Cumrun Vafa, and Shing-Tung Yau. Stringy cosmic strings and noncompact calabi-yau manifolds. *Nucl. Phys.*, B337:1, 1990.

[93] J. Maharana and J. Schwarz. Noncompact symmetries in String Theory. *Nucl. Phys.*, B390:3, 1993.

[94] H. W. Brinkmann. On Riemann Spaces Conformal to Euclidean Space. *Proc. Natl. Acad. Sci. U. S.*, 9:1, 1923.

[95] H. W. Brinkmann. Einstein Spaces which are Mapped Conformally on Each Other. *Math. Annal.*, 94:119, 1925.

[96] Klaus Behrndt, Dieter Lüst, and Wafic A. Sabra. Stationary solutions of $N = 2$ supergravity. *Nucl. Phys.*, B510:264–288, 1998.

[97] Sergio Ferrara, Renata Kallosh, and Andrew Strominger. $N = 2$ extremal black holes. *Phys. Rev.*, D52:5412–5416, 1995.

[98] Andrew Strominger. Macroscopic Entropy of $N = 2$ Extremal Black Holes. *Phys. Lett.*, B383:39–43, 1996.

[99] Sergio Ferrara and Renata Kallosh. Supersymmetry and Attractors. *Phys. Rev.*, D54:1514–1524, 1996.

[100] Sidney R. Coleman, John Preskill, and Frank Wilczek. Quantum hair on black holes. *Nucl. Phys.*, B378:175–246, 1992.

[101] Gregory W. Moore. Arithmetic and attractors. hep-th/9807087, 1998.

[102] Anna Ceresole, R. D'Auria, S. Ferrara, and Antoine Van Proeyen. Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity. *Nucl. Phys.*, B444:92–124, 1995.

[103] Anna Ceresole, R. D'Auria, and S. Ferrara. The Symplectic Structure of $N = 2$ Supergravity and its Central Extension. *Nucl. Phys. Proc. Suppl.*, 46:67–74, 1996.

[104] J. B. Hartle and S. W. Hawking. Solutions Of The Einstein-Maxwell Equations With Many Black Holes. *Commun. Math. Phys.*, 26:87, 1972.

[105] C. Misner. The Flatter Regions of Newman, Unti, and Tamburino's Generalized Schwarzschild Space. *J. Math. Phys.*, 4:924, 1963.

[106] S. D. Majumdar. A Class Of Exact Solutions Of Einstein's Field Equations. *Phys. Rev.*, 72:390, 1947.

[107] A. Papapetrou. A Static Solution of the Equations of the Gravitational Field for an Arbitrary Charge-Distribution. *Proc. Roy. Irish Acad.*, A51:191, 1947.

[108] A. I. Janis, E. T. Newman, and J. Winicour. Reality of the Schwarzschild Singularity. *Phys. Rev. Lett.*, 20:878, 1968.

[109] A. G. Agnese and M. La Camera. General spherically symmetric solutions in charged dilaton gravity. *Phys. Rev.*, D49:2126–2128, 1994.

[110] D. R. Brill and R. W. Lindquist. Interaction Energy in Geometrostatics. *Phys. Rev.*, 131:471–476, 1963.

[111] W. Israel and G. A. Wilson. A Class of Stationary Electromagnetic Vacuum Fields. *J. Math. Phys.*, 13:865, 1972.

[112] J. D. Jackson. The Nature Of Intrinsic Magnetic Dipole Moments. Yellow Report CERN-77-17. Reprinted in V. Stefan and V.F. Weisskopf, eds., Physics and Society: Essays in Honor of Victor Frederick Weisskopf (AIP Press, New York, Springer, Berlin, 1998) 236pp., 1977.

[113] Gerald Kaiser. Distributional sources for Newman's holomorphic field. *J. Phys.*, A37:8735–8746, 2004.

[114] Ezra T. Newman. On a Classical, Geometric Origin of Magnetic Moments, Spin- Angular Momentum and the Dirac Gyromagnetic Ratio. *Phys. Rev.*, D65:104005, 2002.

[115] Andre Gsponer. On the physical interpretation of singularities in Lanczos- Newman electrodynamics. gr-qc/0405046, 2004.

[116] Eric Bergshoeff et al. Superconformal $N = 2$, $D = 5$ matter with and without actions. *JHEP*, 10:045, 2002.